

## TOPOLOGICAL ORDER AND EMERGENCE

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### ABSTRACT

Topologically ordered systems play a prominent role in current research in condensed matter physics; examples include systems that exhibit the quantum Hall effect, topological insulators, and topological superconductors. These systems possess properties that are characterized by topological invariants, exhibit phase transitions that cannot be characterized by spontaneous symmetry breaking, and exhibit order that cannot be characterized in terms of a local order parameter. They thus fall outside the scope of the Landau–Ginsburg theory of phase transitions, which, arguably, has informed much of the discussion, in both the physics and philosophy literature, of emergence in condensed matter systems. Nevertheless, some authors have claimed that topologically ordered systems exhibit emergence. This essay offers a critical assessment of this claim. In particular, it identifies two types of topological order and observes that, whereas the alleged mechanisms underwriting these types differ, they nevertheless share certain features; in particular, the low-energy behavior of such systems can be described by effective topological quantum field theories. This suggests that a unified account of the emergence of topological order should look to a law-centric, as opposed to a mechanism-centric, view of emergence.

# 1. Introduction

Topologically ordered systems play a prominent role in current research in condensed matter physics; examples include systems that exhibit the quantum Hall effect, topological insulators, and topological superconductors (Bernevig and Hughes 2013; Hasan and Kane 2010; Wen 2013). These systems possess properties that are characterized by topological invariants, undergo phase transitions that cannot be characterized by spontaneous symmetry breaking, and exhibit order that cannot be characterized in terms of a local order parameter. They thus fall outside the scope of the Landau–Ginsberg theory of phase transitions, which, arguably, has informed much of the discussion, in both the physics and philosophy literature, of emergence in condensed matter systems. Nevertheless, some authors have claimed that topologically ordered systems exhibit emergence (Chen *et al.* 2010; Lancaster and Pexton 2015; Wen 2013). This essay offers a critical assessment of this claim. In particular, it identifies two types of topological order—symmetry-protected topological order, and intrinsic topological order—and observes that, whereas the alleged mechanisms underwriting these types differ (“short-range entanglement” versus “long-range entanglement”), they nevertheless share certain features; in particular, the low-energy behavior of such systems can be described by effective topological quantum field theories (Qi and Zhang 2011; Qi *et al.* 2008). This suggests that a unified account of the emergence of topological order should look to a law-centric, as opposed to a mechanism-centric, view of emergence. Under a law-centric view, the novelty exhibited by an emergent system with respect to a fundamental system is characterized by distinct laws, as opposed to an underlying mechanism.

In Section 2, I review the Landau–Ginsburg theory of phase transitions and identify three characteristics it attributes to systems that have motivated various authors to describe the latter as exhibiting emergence:

(a) a phase transition, (b) a mechanism (spontaneous symmetry breaking) responsible for this transition, and (c) an effective field theory (EFT) that characterizes the system in the vicinity of a critical point, and that is distinct from the theory that characterizes the system away from this critical point. In Section 3, I consider how topological order differs from Landau–Ginsburg order, and identify two distinct types: symmetry-protected topological order, and intrinsic topological order.<sup>1</sup> Section 4 considers whether systems exhibiting such order can be said to exhibit emergence. My strategy will be to consider the extent to which both types of topologically ordered system exhibit those characteristics that systems described by the Landau–Ginsburg theory exhibit and that have been associated with emergence. Thus, insofar as both types of topologically ordered system exhibit (a) phase transitions, (b) an alleged mechanism responsible for these transitions, and (c) can be characterized by EFTs in the vicinity of their critical points, I claim that they exhibit emergence, at least to the same extent that Landau–Ginsburg systems exhibit emergence. Section 5 then considers two general accounts of emergence, law-centric and mechanism-centric, and suggests that, if emergence is to be attributed to topologically ordered systems, then it is best described by the law-centric view.

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<sup>1</sup> There is no consensus on the terminology used to make this distinction. Some authors make it in terms of "non-interacting" versus "interacting" topological orders, others make it in terms of an "IQHE paradigm" versus a "FQHE paradigm", or "short-range entangled" states versus "long-range entangled" states (Neupert *et al.* 2014, 1). There are also authors who use the term "topological order" only for the second type (Lu and Vishwanath 2012, 1).

## 2. Landau–Ginsburg order and emergence

The central distinction in this essay is between condensed matter systems exhibiting order that is described by what various authors<sup>2</sup> generically call the Landau–Ginsburg theory, on the one hand, and condensed matter systems that exhibit topological order, on the other. This section offers a summary of the Landau–Ginsburg theory and motivations for attributing emergence to systems it describes.

The Landau–Ginsburg theory is concerned with the order exhibited by a physical system that undergoes a continuous (second-order) phase transition from a disordered state to an ordered state. At finite temperatures  $T$ , such a transition is characterized by a non-analyticity in the free energy density  $f$  given by:

$$f = -(1/\beta) \ln Z / V_{space}, \quad Z = \int D\phi_i \exp\{-\beta \int d^d x h[\phi_i]\} \quad (1)$$

where  $V_{space}$  is the volume of  $d$ -dim space, and  $Z$  is the system's partition function. In the latter,  $h[\phi_i]$  is the Hamiltonian density of the system and  $\beta = 1/kT$ , where  $k$  is Boltzmann's constant and  $T$  is the temperature. The Hamiltonian density is a functional of dynamical variables  $\phi_i(x)$ , which in

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<sup>2</sup> See, *e.g.*, Bernivig and Taylor (2013, 1), Hasan and Kane (2010, 3045), Qi and Zhang (2011, 1058), Wen (2004, 5–7, 335–6; 2013, 1–2). By "Landau–Ginsburg theory" (alternatively "Landau paradigm", "Landau theory"), these authors mean the techniques associated with the statistical mechanical approach to continuous phase transitions initiated by Landau and Ginsburg (1950), and extensions of it (*e.g.*, mean field theory, renormalization theory) that allow one to calculate the values of critical exponents.

(1) are considered field variables<sup>3</sup>, and the integral in  $Z$  is over all field configurations. The order associated with such a transition is characterized by a local order parameter  $O[\phi_i]$  that is a functional of the dynamical variables. To say  $O$  is *local* is to say that it depends only on quantities in finite regions of space. To say  $O$  is an *order parameter* is to say that it has a non-zero value in the ordered state, and averages to zero in the disordered state. In the disordered state, while its average is zero, in general it will exhibit non-zero thermal fluctuations.<sup>4</sup> As the critical point in parameter space that represents the phase transition is approached, spatial correlations in the order parameter fluctuations become long-ranged, and at the critical point, the correlation length  $\xi$ , which encodes the typical length of spatial correlations, diverges.<sup>5</sup> This divergence of  $\xi$  is an indication of universality: at a critical point, the system becomes scale-independent insofar as the properties associated with the critical point are independent of the micro-scale properties of the system. Thus many microphysically distinct systems can all possess the same critical point properties.

Another feature of the Landau–Ginsburg theory is its characterization of the ordered phase as breaking a continuous symmetry of the disordered phase. Such symmetry breaking entails, via Goldstone’s theorem, the existence of gapless modes of the system, which

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<sup>3</sup> This assumes the system is near a critical point and hence can be modeled by a continuum theory.

<sup>4</sup> For *classical* phase transitions, thermal fluctuations in the order parameter dominate. For *quantum* phase transitions, quantum fluctuations dominate, and phase transitions can occur at zero temperature. This distinction will be discussed in slightly more detail in Section 4 below.

<sup>5</sup> This ultimately is a reflection of a non-analyticity in the partition function  $Z$ , since the correlation length can be defined in terms of a correlation function, derived from  $Z$ , for the relevant set of quantities.

subsequently allows the construction of a low-energy effective field theory (EFT) of the order parameter fluctuations in the vicinity of the critical point.<sup>6</sup> From this EFT, one can derive values for various critical point properties (*e.g.*, critical exponents). This EFT is a *local* quantum field theory, insofar as its Lagrangian density is a functional of the order parameter fluctuations and their derivatives evaluated at the same point.<sup>7</sup>

To recap so far, in the Landau–Ginsburg framework, phase transitions are characterized by non-analyticities in the free energy density, order is characterized by a local order parameter, a change in order is characterized by spontaneous symmetry breaking, and the low-energy behavior of the system in the vicinity of a critical point can be captured by a local effective field theory. I now turn to the question of how emergence can be attributed to systems that exhibit these characteristics.

### 2.1 Emergence in Landau–Ginsburg Systems

For the purposes of this essay, emergence can be understood as descriptive of the ontology (*i.e.*, entities or properties) associated with a physical system (the emergent system) with respect to another (the fundamental system). It can be minimally characterized by two general criteria, what Crowther (2015) refers to as *Dependence* and *Independence*.

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<sup>6</sup> A gapless mode is a state of the system at energies very close to zero. Such low-energy, or "soft", modes can be considered fluctuations above the ground state energy.

<sup>7</sup> Again, this is a notion of locality as a requirement that the physical quantities of interest must be localized in finite regions of space. The notion of a local order parameter associated with a local QFT in the Landau–Ginsburg theory will be contrasted with the notion of a non-local order parameter associated with a topological QFT in the context of topological order in Section 4.3 below.

Dependence requires the emergent system to be ontologically determined, in some sense, by the fundamental system (it should not “float free” of the latter). Independence requires that the emergent system exhibit a robust sense of novelty with respect to the fundamental system.

One way of motivating the claim that systems that undergo continuous phase transitions exhibit the sort of novelty appropriate for a notion of emergence is suggested by Callender (2001, 549), who considers the following claims:

1. Real systems have a finite number  $N$  of degrees of freedom.
2. Real systems display (continuous) phase transitions.
3. Phase transitions occur when the partition function has a non-analyticity.
4. Phase transitions are described by classical or quantum statistical mechanics.

Callender observes that a problem with these claims is that a non-zero partition function for a finite system cannot display a non-analyticity. In practice, this is addressed by taking the thermodynamic limit, which involves taking  $N \rightarrow \infty$ , while holding  $V/N$  fixed, where  $V$  is the system's volume. One can then show that it is possible for systems with infinite  $N$  to display non-analyticities in their partition functions, and hence exhibit phase transitions; however, this conflicts with claims 1 and 2. Callender suggests that one way to reconcile claims 1–3 is by denying 4: “Statistical mechanics for finite  $N$  is incomplete, unable to describe phase transitions; therefore, they are in some sense emergent” (Callender 2001, 549). Thus to deny claim 4 is to say an ordered system that is the result of a phase transition from a disordered system, is novel with respect to the latter in the sense that the transition cannot be completely described within the statistical mechanical (*viz.*, Landau–Ginsburg) framework.

Some authors adopt what I will call a mechanism-centric view of emergence. This view attributes a mechanism to emergent phenomena which is supposed to provide the causal/mechanical explanation of how novelty arises, and thus how Dependence can be (causally/mechanically) reconciled with Independence. Without such a mechanism, it is claimed, the notion of emergence risks becoming trivial: “...emergent properties are not a panacea, to be appealed to whenever we are puzzled by the properties of large systems. In each case, we must produce a detailed physical mechanism for emergence, which rigorously explains the qualitative difference that we see with the microphysical” (Mainwood 2006, 284). Mainwood (2006, 107) associates this appeal to a physical mechanism with the “new emergentism” of prominent condensed matter physicists (*e.g.*, Anderson 1972; Laughlin and Pines 2000), and, in the context of the Landau–Ginsburg theory, identifies the mechanism of most interest as spontaneous symmetry breaking: “The claim of the New Emergentists is that in the phenomenon of symmetry-breaking we have a mechanism by which the set of ‘good coordinates’ of the whole can be entirely different from the sets of good coordinates which apply to the constituent parts when in isolation or in other wholes”. Morrison (2012, 148) concurs: “...understanding emergent phenomena in terms of symmetry breaking—a structural dynamical feature of physical systems...—clarifies both how and why emergent phenomena are independent of any specific configuration of their microphysical base.”

The notion of a mechanism can be understood in two ways. A microphysical mechanism can be thought of as a collection of entities and processes that realize a general principle or regularity (Weber *et al.* 2013, 59). Alternatively, a high-level mechanism can be thought of as a general physical process that can be realized by any number of concrete microphysical mechanisms. Examples of the latter include Morrison’s (2012, 149) “structural/dynamical feature of physical systems” and Laughlin and Pine’s (2000, 28) “higher organizing principle”. Advocates of high-level mechanisms point to multiple realizability as an essential



feature of emergence: the instantiation of a high-level mechanism by many distinct microphysical mechanisms that all share the same general (viz., “universal”) features is taken as a sign that these features possess a robust sense of novelty and hence can be taken to be emergent.

An alternative view of emergence is what I will call a law-centric view. According to this view, the novelty associated with an emergent system is underwritten, not by an appeal to an underlying mechanism, but rather by an appeal to distinct laws. According to Bain (2013), this is exemplified by physical systems whose low-energy behavior can be described by an effective field theory (EFT). Such a system is described, at high-energies, by a Lagrangian density  $\mathcal{L}[\phi]$  that is a functional of dynamical variables  $\phi$ . At low-energies, the system can be described by an effective Lagrangian density  $\mathcal{L}_{\text{eff}}[\theta] \neq \mathcal{L}[\phi]$  that is a functional of a different set of dynamical variables  $\theta$ , and that is formally distinct from  $\mathcal{L}$ . Since the low-energy behavior of the system is governed by laws, encoded in  $\mathcal{L}_{\text{eff}}[\theta]$ , and dynamical variables  $\theta$  that are formally distinct from the laws, encoded in  $\mathcal{L}[\phi]$ , and dynamical variables  $\phi$  that govern its high-energy behavior, the low-energy behavior is dynamically independent of, and dynamically robust with respect to, the high-energy behavior, and hence is novel with respect to the latter. In the context of the Landau–Ginsburg framework, this suggests that since a system in the vicinity of a critical point can be characterized by an EFT that is formally distinct (both in terms of laws and in terms of dynamical variables) from the theory that describes the system away from the critical point, the system near the critical point is dynamically independent of, and dynamically robust with respect to, the system away from the critical point; hence the former is emergent with respect to the latter.

Thus, the claim that Landau–Ginsburg systems exhibit emergence can be underwritten by appealing to the following characteristics such systems display:

- (a) Landau–Ginsburg systems exhibit phase transitions;
- (b) Landau–Ginsburg systems exhibit a mechanism (spontaneous symmetry breaking) responsible for (a); and/or,
- (c) Landau–Ginsburg systems can be described by an effective field theory (EFT) that characterizes the system in the vicinity of a critical point, and that is distinct from the theory that characterizes the system away from the critical point

### 3. Topological order

As noted in Section 2, in the Landau–Ginsburg framework, a phase transition is characterized by a non-analyticity in the free energy density, order is characterized by a local order parameter, a change in order is characterized by a spontaneously broken symmetry, and the low-energy behavior of a system in the vicinity of a critical point is described by an effective field theory (EFT) that takes the form of a local quantum field theory. In a topologically ordered system, a phase transition is characterized by a non-analyticity in the ground state energy density, order is characterized by a non-local order parameter (a topological invariant), a change in order is not characterized by a spontaneously broken symmetry, but rather a change in the topology of the state space, and the low-energy behavior of the system in the vicinity of a critical point is described by an EFT that takes the form of a topological quantum field theory. In this section and the next I will attempt to explain these differences in slightly more detail. This section reports on two distinct types of topological order: symmetry-protected topological order, and intrinsic topological order. Section 4 returns to the question of whether such systems can be said to exhibit emergence.

### 3.1 Symmetry-Protected Topological (SPT) Order

The first type of topological order is referred to in the physics literature as symmetry-protected topological order. The nature of systems exhibiting it can be understood within the framework of the band theory of solids (see, *e.g.*, Hasan and Kane 2010). According to this theory, electrons in a 2-dim crystal are described by Bloch states  $|u_m(\mathbf{k})\rangle$  characterized by energy levels  $m$  and periodic momenta  $\mathbf{k}$  restricted to a closed unbounded region (the “Brillion zone”) in momentum space  $\mathcal{P}$ . An insulator is characterized by an energy gap that separates the occupied valence-band electron states from the empty conduction-band states. The state space for an insulator can be represented by a vector bundle, call it  $V$ , over the base space  $\mathcal{P}$ , with typical fiber given by a Hilbert space  $\mathcal{H}$  of Bloch states (the latter are then represented by sections of  $V$ ).<sup>8</sup> Geometrically,  $V$  can be flat or curved, with curvature represented by the Berry curvature  $\mathcal{F}_m = \nabla \times A_m$ , where  $A_m = i\langle u_m | \nabla_{\mathbf{k}} | u_m \rangle$  is a connection defined on  $V$ .<sup>9</sup> As a vector bundle,  $V$  is locally isomorphic to the direct product space  $\mathcal{H} \times \mathcal{P}$ . Globally, however,  $V$  may be “non-trivial” in the sense that its fibers may be “twisted”.<sup>10</sup> Such twisting can be encoded in a topological invariant called the first Chern number  $n_m$ , defined by

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<sup>8</sup> A vector bundle consists of a set of vector spaces (the “fibers” of the bundle) that are parameterized by the points of a base space in such a way that the fibers are smoothly related to each other.

<sup>9</sup> The Berry curvature characterizes the behavior of a Bloch state upon parallel transport around a closed curve  $C$  in  $\mathcal{P}$ . A change in the state is reflected in a non-trivial Berry phase, defined by  $\gamma_m = \int_C A_m dk$ .

<sup>10</sup> Recall that such twisting of fibers is one way to characterize a Möbius strip as a fiber bundle over the circle  $S^1$ , with the unit interval  $[0, 1]$  as typical fiber.

$$n_m = (1/2\pi) \int_{\text{BZ}} \mathcal{F}_m d^2\mathbf{k} \quad (2)$$

where the integral is over all electron momenta in the Brillouin zone.<sup>11</sup> The total Chern number is then given by summing  $n_m$  over all  $N$  occupied energy bands:

$$n = \sum_{m=1}^N n_m \quad (3)$$

It now transpires that, while conventional insulators are characterized by a trivial (*i.e.*,  $n = 0$ ) state space bundle structure, there are insulating systems with a state space characterized by non-trivial total Chern numbers. The paradigm case of this is the integer quantum Hall effect (IQHE). This effect occurs when a 2-dim conductor is placed in an external magnetic field perpendicular to its surface. At low temperatures and high magnetic fields, the transverse (or ‘‘Hall’’) conductivity  $\sigma_H$  becomes quantized in units of  $e^2/h$ ,

$$\sigma_H = p(e^2/h) \quad (4)$$

for integer  $p$ . Thouless, *et al.* (1982) showed that the Hall conductivity can be expressed by

$$\sigma_H = (e^2/h) \sum_{m=1}^N (1/2\pi) \int_{\text{BZ}} \mathcal{F}_m d^2\mathbf{k} \quad (5)$$

thus identifying the integer  $p$  with the total Chern number of the state space. Different values of  $p$  correspond to topologically distinct state spaces that cannot be transformed into each other by local deformations,

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<sup>11</sup> The first Chern number is an instance of the Gauss–Bonnet theorem  $\chi_M = (1/2\pi) \int_M K dS$ , where  $M$  is a 2-dim closed manifold without boundary,  $K$  is the local Gaussian curvature of  $M$ , and  $\chi_M$  is an integer known as the Euler characteristic of  $M$  (Nakahara 2003, 462.) It can also be expressed by  $\chi_M = 2(1 - g)$ , where the genus  $g$  of  $M$  is the number of its handles.

so long as the essential gapped structure of the energy spectrum is preserved. This suggests that distinct values of  $p$  correspond to distinct IQHE phases. Notably, such phases are not characterized by different symmetries, hence transitions between phases cannot be described by spontaneous symmetry breaking. Each phase consists of a gapped bulk insulator state and gapless conducting edge states characterized by a given value of  $\sigma_H$ . The conducting states are “topologically protected” in the sense that they are robust under local deformations of the state space that preserve its topology. Such a system that is an insulator in the bulk but a topologically protected conductor at its edge is referred to as a “topological insulator”.

In another type of topological insulator, the role played by the external magnetic field in the IQHE is played by spin-orbit coupling, and whereas IQHE systems are not time-reversal invariant (due to the external magnetic field), so-called quantum spin Hall effect (QSHE) systems are. Kane and Mele (2005) demonstrated that such systems can be characterized by a  $Z_2$  topological invariant of their state spaces. This characterization was subsequently extended to 3-dimensional QSHE systems. Topological insulators that exhibit the QSHE differ from those that exhibit the IQHE insofar as the former have symmetries (typically time-reversal invariance), and the conducting states are robust only under local deformations of  $\mathcal{P}$  that preserve these symmetries. They are thus referred to as exhibiting *symmetry-protected topological* (SPT) order.

SPT order is similar to Landau–Ginsburg order insofar as both are characterized, in part, by symmetries; however, transitions between SPT orders are *not* characterized by a spontaneously broken symmetry; on the contrary, the relevant symmetry is preserved under an SPT transition. Note finally, that IQHE systems are not characterized by symmetries, and hence, perhaps, should not be considered as possessing SPT order. For the purposes of this essay, however, it is convenient to group both under the same category. One might consider an IQHE system to be a trivial example of SPT order, for instance, where the relevant

symmetry is just the identity. More importantly, both IQHE systems and SPT ordered systems are one-body “non-interacting” topological insulators, insofar as they are completely described by a one-body Hamiltonian in which electron–electron interactions are ignored.<sup>12</sup> This is a fundamental feature that distinguishes these types of topologically ordered systems from the second type.

### 3.2 *Intrinsic Topological Order*

The second type of topological order is referred to as intrinsic topological order.<sup>13</sup> Systems that exhibit this type of order have the characteristics of topological insulators; namely, they are gapped systems that possess a bulk insulating state and robust conducting edge states.<sup>14</sup> This robustness is not the result of a one-body interaction with an external magnetic field, or internal spin-orbit coupling, as in SPT ordered systems;

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<sup>12</sup> Ryu *et al.* (2010) provide an exhaustive classification of non-interacting topological insulators (and related systems known as topological superconductors) in all spatial dimensions, based on whether they possess an integer ( $Z$ ) topological invariant (like the IQHE), or a  $Z_2$  topological invariant, and on the type of symmetry (if any) that is topologically protected. The latter include time-reversal  $\mathcal{T}$ , charge-conjugation  $\mathcal{C}$ , and the product  $\mathcal{S} = \mathcal{TC}$ , referred to as a chiral, or sublattice, symmetry. Since this “tenfold way” includes IQHE systems, it seems appropriate in this context to group the latter with SPT ordered systems in general.

<sup>13</sup> This terminology follows Chen *et al.* (2010). Neupert *et al.* (2014, 1) report that this second type appears in the physics literature under various names, including “fractional topological insulators, long-range entangled phases, topologically ordered phases, or symmetry enriched topological phases”.

<sup>14</sup> Bernevig and Taylor (2013, 3) report that there are exceptions.

rather, it is a many-body effect due to electron–electron interactions.<sup>15</sup> It turns out that there are two consequences of this more complex type of interaction. First, unlike SPT ordered systems, intrinsic topologically ordered systems are characterized by degenerate ground states. Second, unlike SPT ordered systems, intrinsic topologically ordered systems possess bulk excitations that take the form of quasiparticles and that obey fractional statistics.

The “paradigm” of an intrinsic topologically ordered system is one that exhibits the fractional quantum Hall effect (FQHE). The FQHE occurs for fractional values of  $\nu$  in equation (4). Thus the identification of  $\nu$  with the (integer) first Chern number is no longer possible, and one has to look elsewhere for a topological invariant to explain robustness. Wen (2013, 1990) suggests that the topological nature of FQHE systems is encoded in the measures of the two features that distinguish them from IQHE systems (and other SPT ordered systems); namely:

- (i) The degeneracy of the ground state of such systems depends on the genus of the parameter space.
- (ii) The fractional statistics exhibited by the bulk excitations of such systems are encoded in non-Abelian Berry phases of the degenerate ground states.

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<sup>15</sup> This description in terms of a many-body effect of electrons is sufficient for the purpose of making an initial distinction between intrinsic topological order and SPT order; however, the former can also be described in terms of a one-body effect of composite particles (fermions or bosons), as Section 5 will briefly report.

Other systems that can be characterized by these measures include chiral spin liquids and  $Z_2$  spin liquids (Chen *et al.* 2010, 1–2).<sup>16</sup>

## 4. Do topologically ordered systems exhibit emergence?

Numerous authors have suggested that topologically ordered systems exhibit emergence. With respect to intrinsic topologically ordered systems, Wen (2013, 6, 11) refers to a “principle of emergence” and the “emergence of fractional statistics and topological degeneracy on compact spaces”. Elsewhere he claims “[t]opological order has many new emergent phenomena, such as emergent gauge theory, fractional charge, fractional statistics, non-Abelian statistics, and perfect conducting boundary” (2013, 16). Moore (2010, 197) refers to “...the emergence of quasiparticles with modified charge and statistics” in FQHE states, and Hasan and Kane (2010, 3050) cite “...the emergence of Majorana fermions in [topological] superconducting systems”. Similarly, Qi and Zhang (2011, 1104) refer to “...the emergence of the topological superconducting phase at a quantum Hall plateau transition.” Finally, Lancaster and Pexton (2015, 343) seek to make sense of emergence in FQHE systems in terms of the mechanism of long-range entanglement, which they claim underwrites an “intrinsic holism”, and because of this, “...the FQHE bears serious consideration as an example of a metaphysically significant, ‘strongly’ emergent phenomenon”.

Note that these claims attribute emergence to *both* types of topological order: SPT order (which includes topological

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<sup>16</sup> It perhaps should be noted that the only types of intrinsic topologically ordered system that have been experimentally realized are FQHE systems (thanks to a referee for making this explicit).



superconductors), and intrinsic topological order (which includes phenomena, like the FQHE, associated with ground state degeneracies and bulk fractional statistics). In assessing these claims, I will take my cue from the earlier discussion of emergence in Landau–Ginsburg systems. While both types of topologically ordered system differ in fundamental respects from Landau–Ginsburg systems, the relevant question is whether or not they share those features of Landau–Ginsburg systems that have motivated the claim that the latter exhibit emergence. This question comes in three parts:

- (a) Can a topologically ordered system be described as the result of a phase transition?
- (b) Can a mechanism be identified that underwrites such a transition?
- (c) Can a topologically ordered system be characterized in the vicinity of a critical point by an effective field theory (EFT) that is distinct from the theory that describes the system away from the critical point?

In anticipation of the subsequent discussion, the answers to all of these questions will be “yes”.

#### 4.1 *Phase Transition*

In a Landau–Ginsburg ordered system, the transition that separates distinct phases can either be a classical or a quantum phase transition. In this context, a classical phase transition is one in which thermal fluctuations in the order parameter dominate, whereas a quantum phase transition is one in which quantum fluctuations dominate, with the result that a transition between phases can occur at zero temperature. Thus, while classical phase transitions are characterized by non-analyticities in the free energy density  $f$  defined in equation (1), quantum phase transitions are characterized by non-analyticities in the ground state energy density  $\rho_E$ . The latter is given by

$$\rho_E = i(\ln Z / V_{\text{spacetime}}), \quad Z = \int D\phi_i \exp\left\{ \int d^d x d\tau \mathcal{L}[\phi_i] \right\} \quad (6)$$

where  $V_{\text{spacetime}}$  is the volume of spacetime, and in the partition function  $Z$ ,  $\mathcal{L}[\phi_i]$  is the Lagrangian density of the quantum system (see, *e.g.*, Wen 2004, 350; Vojta 2003, 2078). In equation (6), the partition function  $Z$  for a quantum system in  $d$  dimensions has been identified with a Feynman integral in imaginary time  $\tau$ , in  $d + 1$  spacetime dimensions (the latter, formally, can be regarded as the partition function for a classical system in  $d + 1$  dimensions).<sup>17</sup>

In a topologically ordered system, the expectation is that the transition between phases takes the form of a quantum phase transition. In this context, this is a transition that occurs at zero temperature and is characterized by a non-analyticity in the ground state energy density, but need not necessarily be thought of as dominated by quantum fluctuations in the order parameter (insofar as there may not be a readily identifiable local order parameter). Another way to characterize quantum phase transitions that addresses this difference and that will be relevant in the subsequent discussion of mechanism is given by Chen *et al.* (2010, 3). Let  $H(g)$  be a Hamiltonian with a smooth dependence on some parameter  $g$ . This induces a dependence  $\langle O \rangle(g)$  on the ground state expectation value of a local operator  $O$ . The quantum system described by  $H(g)$  can be said to undergo a quantum phase transition at  $g = g_c$  just

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<sup>17</sup> See Vojta (2003, 2076) for a discussion of this "quantum-classical mapping". One way to motivate it is by observing that the term  $\exp\{-H/kT\}$ , with Hamiltonian  $H = \int d^d x h[\phi]$  that appears in the classical partition function in equation (1) resembles a time evolution operator if one identifies the time parameter as  $1/kT = \tau = -it/\hbar$ , where  $t$  is a real time parameter. At a zero temperature critical point, this imaginary time parameter acts like a spatial dimension insofar as it becomes infinite, signifying that the system is infinite.

when the function  $\langle O \rangle(g)$  has a non-analyticity at  $g_c$  in the thermodynamic limit. A quantum phase can then be defined as an equivalence class of ground states  $|\Phi(g)\rangle$  of all Hamiltonians that can be connected by a smooth path in the associated parameter space. In other words, two ground states  $|\Phi(0)\rangle, |\Phi(1)\rangle$ , of  $H(0)$  and  $H(1)$ , respectively, belong to the same quantum phase just when there is a smooth path of Hamiltonians  $H(g), 0 \leq g \leq 1$ , that connects  $H(0)$  and  $H(1)$  and that does not contain a quantum phase transition.

#### 4.2. Mechanism

The mechanism associated with transitions between Landau–Ginsburg phases is spontaneous symmetry breaking. Chen *et al.* (2010, 4) suggest that the mechanism associated with transitions between intrinsic topologically ordered phases be identified with what they call *long-range entanglement*.<sup>18</sup> This is defined in terms of the following:

a state has only short-range entanglement [SRE] if and only if it can be transformed into an unentangled state (*i.e.*, a direct-product state) through a local unitary evolution. If a state cannot be transformed into an unentangled state through a LU [local unitary] evolution, then the state has long-range entanglement (LRE). (Chen *et al.* 2010, 4.)

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<sup>18</sup> Recall from Section 2 that advocates of a mechanism-centric view of emergence consider spontaneous symmetry breaking (SSB) as a high-level, as opposed to microphysical, mechanism and stress its multiple realizability as underwriting the type of novelty that can be associated with emergence. The literature on long-range entanglement, on the other hand, when it makes this distinction, suggests that long-range entanglement is a microphysical mechanism (see, *e.g.*, Wen 2013, 14).

A “local unitary evolution” is a unitary operation generated by a local Hamiltonian over a finite time. Intuitively, an LRE state is a state that cannot be disentangled by local dynamical operations; *i.e.*, operations encoded in a local Hamiltonian.

There seem to be two motivations for identifying long-range entanglement as the mechanism for intrinsic topologically ordered phases. The first is based on the claim that two gapped states belong to the same phase if and only if they are related by a local unitary evolution (Chen *et al.* 2010, 3). In the jargon of the previous subsection, suppose the ground states  $|\Phi(0)\rangle$ ,  $|\Phi(1)\rangle$  of Hamiltonians  $H(0)$  and  $H(1)$  are gapped. The claim then is that there is an adiabatic path  $H(g)$  that connects  $H(0)$  and  $H(1)$  (*i.e.*, the ground states belong to the same phase) if and only if

$$|\Phi(1)\rangle = U|\Phi(0)\rangle, \quad U = T\left[\exp\left\{i\int_0^1 dg \tilde{H}(g)\right\}\right] \quad (7)$$

where  $U$  is a local unitary evolution in which  $T$  is the path-ordering operator, and  $\tilde{H}(g) = \sum_i O_i(g)$  is a sum of local operators (technically,  $\tilde{H}(g)$  need not take the same form as the Hamiltonian  $H(g)$ , but the intent is that it encode the same local information as the latter). Thus SRE gapped states belong to phases associated with unentangled, direct-product states, whereas LRE gapped states do not.

The second motivation for identifying LRE as the mechanism for intrinsic topologically ordered phases seeks to identify direct-product states with “trivial topology” (*i.e.*, the trivial case of intrinsic topological order). This motivation doesn’t appear explicitly in the literature.<sup>19</sup> It can be reconstructed in terms of two steps:

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<sup>19</sup> Chen *et al.* (2010, 4), for instance, simply state: “Since a direct-product state is a state with trivial topological order, we see that a state with a short-range entanglement also has a trivial topological order.” The motivation here

1. The first step is based on the fact that the bulk states of intrinsic topologically ordered phases obey fractional statistics. Technically, this means that these states carry representations, of  $\dim > 1$ , of the braid group  $B_N$  on  $N$  elements; one says they obey “non-Abelian braiding statistics”.<sup>20</sup>

The condition that the braid group  $B_N$  acts on the Hilbert space of states says nothing about whether the states that carry representations of  $B_N$  are entangled, let alone about whether the states that carry the “trivial” representation of  $B_N$  are direct-product states. An additional step seems to be necessary to support these inferences.

2. The second step is based on an analogy between the non-locality of states that obey non-Abelian braiding statistics, on the one hand, and the non-locality of entangled states on the other. Kauffman and Lomonaco (2002, 2) refer to the former as “topological entanglement” and suggest, “a topological entanglement is a non-

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apparently comes from fault-tolerant quantum computation, in which states that obey non-Abelian braiding statistics are proposed as the basis for a “topological” quantum computer. In this context, Bravyi *et al.* (2006, 3) identify a concept of “topological quantum order” (TQO) as a property of a ground state just when it cannot be distinguished or mapped into another orthogonal ground state by means of local operations (intuitively, two such ground states are distinct yet share the same local properties). But there is no essential relation between this type of nonlocality and the nonlocality associated with non-Abelian braiding statistics.

<sup>20</sup> See, *e.g.*, Nayak *et al.* (2008, 1085). Upon parallel transport around a closed curve, such a state picks up a nontrivial Berry phase  $|\psi\rangle \rightarrow e^{i\theta}|\psi\rangle$ ,  $0 \leq \theta < 2\pi$ . The special cases  $\theta = 0, \pi$  correspond to Bose–Einstein and Fermi–Dirac statistics, respectively.

local structure of a topological system”, whereas “a quantum entanglement is a non-local structural feature of a quantum system”. They support this analogy with examples of non-trivial topologically entangled states that represent links and braids, on the one hand, and corresponding entangled states of quantum systems on the other: for instance, cutting a component of a link removes its “topological entanglement”, just as measuring a quantum state removes its entanglement.<sup>21</sup> Based on this analogy, an unentangled (direct-product) state is analogous to a state with trivial braiding topology.<sup>22</sup>

Taken together, these two motivations suggest that all SRE gapped states belong to the same phase as a state with trivial topological order (no “topological entanglement”, or trivial non-Abelian braiding statistics). An LRE gapped state, on the other hand, belongs to a phase with non-trivial topological order; and, in principle, not all LRE gapped states belong to the same such phase. Different LRE states can belong to different phases, each distinguished by a particular topological order (*viz.*, topological entanglement, or non-Abelian braiding statistics). This identification of LRE states with intrinsic topological order leaves unanswered the question of how SPT order should be characterized. Chen *et al.* (2010, 5–6) answer this question by distinguishing cases of

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<sup>21</sup> Kauffman and Lomonaco (2002, 3) are careful to point out that the analogy is basis dependent: a given entangled state decoheres in different ways, depending on the chosen basis; or, as they put it, “From a physical standpoint, seeing the [entangled] state as analogous to a link depends upon the choice of an observable”.

<sup>22</sup> Kauffman and Lomonaco (2002, 7) caution that “...the question of the precise relationship between topological entanglement and quantum entanglement certainly awaits the arrival of more examples of unitary representations of the braid group”.

local unitary evolutions (7) in which the Hamiltonian does not possess symmetries, from cases in which it does. For the former cases, all SRE states belong to the same topologically trivial (*viz.*, direct-product) phase. However, for the latter cases, equivalence classes of phases divide into two groups: those generated by local unitary evolutions that break a relevant symmetry, and those that preserve a relevant symmetry:

- (i) SRE states of symmetric gapped systems related by a local unitary evolution that break a relevant symmetry can belong to different phases, depending on which symmetry is broken. Chen *et al.* identify this type of SRE state as a Landau–Ginsburg phase.
- (ii) SRE states of symmetric gapped systems related by a local unitary evolution that preserves a relevant symmetry can belong to different phases, depending on which symmetry is preserved. Chen *et al.* identify this type of SRE state as an SPT phase.<sup>23</sup>

To summarize the discussion so far, Chen *et al.* (2010) suggest that the mechanism responsible for intrinsic topological order is long-range entanglement (either symmetry-preserving or symmetry-breaking), whereas the mechanism responsible for SPT order is symmetry-preserving short-range entanglement.

#### 4.3. *Effective Field Theory*

Recall that Landau–Ginsburg systems are characterized, in part, by an effective field theory (EFT) at a critical point that takes the form of a local

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<sup>23</sup> To be complete, there are also LRE states of symmetric gapped systems that can belong to different phases depending on which symmetry the defining local unitary evolution breaks or preserves. Examples of the former are topological superconductors, and examples of the latter are  $Z_2$  symmetric spin liquids (Chen *et al.* 2010, 6).

quantum field theory. A topologically ordered system can likewise be characterized by an EFT at a critical point, but the latter takes the form of a *topological* quantum field theory (TQFT). This distinction can be made in the following way (after Labastida and Lozano 1997, 5). In the Feynman integral approach, a local quantum field theory consists of a smooth  $(d+1)$ -dimensional manifold  $M$  (*i.e.*, spacetime), a Lorentzian metric  $g_{\mu\nu}$  on  $M$ , a set of fields  $\{\phi_i(x)\}$ , and a Lagrangian density  $\mathcal{L}[\phi_i]$  that is a functional of the fields and their derivatives evaluated at the same point. The measurable observables are vacuum expectation values of products of local operators  $O[\phi_i]$  constructed as functionals of the fields. These are defined with respect to the Feynman path integral via:

$$\langle O_1 \dots O_n \rangle = (1/Z) \int D\phi_i O_1 \dots O_n \exp\{i \int d^{d+1}x \mathcal{L}[\phi_i]\} \quad (8)$$

where  $Z$  is the integral in (6) (with real time replacing imaginary time). In this approach, a topological quantum field theory is a local quantum field theory in which

$$\delta/\delta g^{\mu\nu} \langle O_1 \dots O_n \rangle = 0 \quad (9)$$

for some set of local operators. In words: The vacuum expectation value of these operators is invariant under variations of the metric. One way to guarantee this is if the terms in the Lagrangian density involving these operators are independent of the metric.<sup>24</sup> This independence can be

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<sup>24</sup> TQFTs with this property are referred to as Schwarz type (Labastida and Lozano 1997, 5). Another way to secure (9) is what is called a cohomological TQFT, or a TQFT of Witten type (Labastida and Lozano 1997, 7). It should also be noted that, in addition to the above approach to TQFTs, there is an axiomatic approach that defines a TQFT as a functor from the category **nCob** of  $n$ -dimensional cobordisms to the category **Hilb** of finite-dimensional Hilbert spaces (Baez 2006, 248).



understood as a non-local property insofar as it indicates that the vacuum expectation value is a topological invariant of  $M$ .<sup>25</sup>

The Lagrangian density that encodes a TQFT is local insofar as it is a functional of fields (and their derivatives) that are evaluated at a single spacetime point. A TQFT is non-local insofar as it possesses observables that are metric-independent in the sense of (9). The non-local order parameter that appears in the EFT for a topologically ordered system is such a non-local observable. This should be contrasted with the local order parameter that appears in the EFT for a Landau–Ginsburg ordered system, which can be defined as a metric-dependent observable.

An example of an EFT for an intrinsic topologically ordered system is the TQFT that describes a system that exhibits the FQHE. This TQFT is encoded in the effective Lagrangian density,

$$\mathcal{L}_{\text{eff}}[a_\mu, A_\mu, j^\mu] = -\frac{s}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \frac{e}{2\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda + j^\mu a_\mu, \quad (10)$$

$\mu, \nu, \lambda = 0, 1, 2$

where  $s$  is an odd integer, the first term encodes the dynamics of a Chern–Simons potential gauge field  $a_\mu$  and the second and third terms encode the coupling of a magnetic potential  $A_\mu$  and a source  $j^\mu$  of quasiparticles to the Chern–Simons potential, respectively (see, *e.g.*, Wen 2004, 298). Note that the first two terms in (10) are independent of a spacetime metric (contraction of indices is performed using the totally antisymmetric tensor  $\epsilon^{\mu\nu\lambda}$ ).

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<sup>25</sup> Strictly speaking, this notion of "topological invariant" is weaker than the notion that appears in topology (see, *e.g.*, Birmingham *et al.* 1991, 136). In the latter, a topological invariant of a manifold  $M$  is a quantity that is constant on the space of homeomorphism equivalence classes of  $M$ . A metric-independent quantity is constant on the smaller space of all metrics on  $M$ .

The effective Lagrangian density (10) is intended to describe the behavior of a system that exhibits the FQHE in the (low-energy) vicinity of a critical point. The theory of the system away from the critical point is described by a formally distinct Lagrangian density:

$$\mathcal{L}[\psi, A_\mu] = \psi^\dagger i(\partial_i - iA_i)\psi + \frac{1}{2m} \psi^\dagger (\partial_i - iA_i)^2 \psi + V(\psi^\dagger, \psi),$$

$$i = 1, 2 \quad (11)$$

where  $\psi$  is a second-quantized electron field,  $m$  is the electron mass, and the last term encodes the Coulomb interaction between electrons. The Lagrangian density (11) is intended to describe the behavior of electrons in a 2-dim conductor in the presence of an external magnetic field at temperatures and magnetic field strengths away from the quantum critical point. Note that (11) is a local quantum field theory that is not a TQFT; in particular, the terms in (11) all depend on spatial and temporal metrics (the Lagrangian (11) encodes a non-relativistic local quantum field theory). In this case we have  $\mathcal{L}_{eff}[a_\mu, A_\mu, j^\mu] \neq \mathcal{L}[\psi, A_\mu]$ , and the dynamical variables  $(a_\mu, A_\mu, j^\mu)$  of the EFT are distinct from those  $(\psi, A_\mu)$  of the high-energy theory.

Qi *et al.* (2008) (see also Qi and Zhang 2011) show how effective TQFTs similar to (10) can be constructed for SPT ordered systems near critical points in all the relevant dimensions. These systems can also be characterized by non-relativistic local quantum field theories similar to (11) away from critical points. An example of an EFT for an SPT ordered system is the TQFT that describes a system that exhibits the integer quantum Hall effect (IQHE). This TQFT is encoded in the (2+1)-dimensional effective Lagrangian density,

$$\mathcal{L}_{eff}[A_\mu] = \frac{C_1}{4\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \quad \mu, \nu, \lambda = 0, 1, 2 \quad (12)$$

where  $C_1$  is an integer (identified as the first Chern number; Qi *et al.* 2008, 3). In (12), the low-energy degrees of freedom are encoded in a non-interacting electromagnetic potential  $A_\mu$  alone, as opposed to an additional Chern–Simons field  $a_\mu$ .<sup>26</sup> An extension of (12) exists for the IQHE in (4+1)-dimensions:

$$\mathcal{L}_{eff}[A_\mu] = \frac{C_2}{24\pi} \epsilon^{\mu\nu\rho\sigma} A_\mu \partial_\nu A_\rho \partial_\sigma A_\tau, \quad (13)$$

$$\mu, \nu, \rho, \sigma, \tau = 0, 1, 2, 3, 4$$

where  $C_2$  is an integer (identified as the second Chern number; Qi *et al.* 2008, 11). The effective Lagrangian (13) was first written down by Bernevig *et al.* (2002) in their field-theoretic formulation of Zhang and Hu's (2001) four spatial dimensional extension of the quantum Hall effect. Whereas a QHE system in two spatial dimensions breaks time-reversal symmetry (due to the external magnetic field), in four spatial dimensions, it turns out, the system is time-reversal invariant. Qi *et al.* take (12) to be the general form of the Lagrangian density for a time-reversal broken (TRB) SPT ordered system in (2+1)-dim, and obtain Lagrangian densities for TRB SPT ordered systems in (1+1)-dim and (0+1)-dim *via* a process of dimensional reduction. They take (13) to be the general form of the Lagrangian density for a time-reversal invariant (TRI)

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<sup>26</sup> Qi (2013, 95) calls the theory encoded in (12) a "topological response theory", which is used to describe perturbations of the system due to the external potential  $A_\mu$ . In contrast, Qi (2013, 96) calls the theory encoded in (10) a "dynamical topological field theory" insofar as it describes dynamical "topological" degrees of freedom of the system, as encoded in the Chern–Simons field  $a_\mu$ , coupled to the external probe field  $A_\mu$ . One can show that the transverse (*i.e.*, Hall) conductivity derived from (10) is given by  $\sigma_H = (1/s)(e^2/h)$ , whereas for (12), it is given by  $\sigma_H = C_1 e^2/h$ . (Note that a hierarchical extension of (10) can be constructed for fractional filling factors other than  $1/s$  (Wen 2004, 301).)

SPT ordered system in (4+1)-dim, and obtain Lagrangian densities for TRI SPT ordered systems in (3+1)-dim and (2+1)-dim *via* dimensional reduction.

The above discussion indicates that both SPT ordered systems and intrinsic topologically ordered systems exhibit phase transitions, and these phase transitions can be associated with mechanisms, as well as EFTs in the vicinity of critical points (where the latter are simply associated with the phase transitions expected to separate distinct phases of topologically ordered systems). Thus, to the extent that emergence is ascribed to Landau–Ginsburg phases that possess these characteristics, it should also be ascribed to both types of topologically ordered system. The question for the next section is, of the two views of emergence, mechanism-centric and law-centric, which suggests itself more in the context of topological order?

## 5. Mechanism-centrism versus law-centrism

In general, the task of articulating a notion of emergence is to resolve the tension between the dependence of an emergent system on a fundamental system, and its independence from the latter. A mechanism-centric view of emergence resolves this tension by positing a mechanism that is responsible for independence in the presence of dependence. A law-centric view underwrites independence by an appeal to distinct laws. In this section, I argue that a mechanism-centric view faces trouble in the context of topologically ordered systems.

On the one hand, Section 4 argued that systems that exhibit SPT order and intrinsic topological order should be ascribed emergence, at least to the extent that emergence is ascribed to systems that exhibit Landau–Ginsburg order. On the other hand, if emergence requires an appeal to a physical mechanism, and if the mechanisms for both types of topological

order are identified with SRE and LRE, respectively, then potential trouble awaits. Recall from Section 4.2 that a state is SRE *if and only if* it can be transformed into a direct-product state *via* a local unitary evolution. A state is LRE *if and only if* it cannot be transformed into a direct-product state *via* a local unitary evolution, and hence *if and only if* it is not SRE. Thus, naively, if LRE is a mechanism for emergence, then SRE cannot be a mechanism that produces the same type of emergence. A mechanism-centric advocate of emergence is thus faced with the following options:

- (a) Reconcile the two types of topological order and their contrasting mechanisms of SRE and LRE, on the one hand, with either a common notion of emergence, or at least with distinct notions that are compatible; or,
- (b) Deny that emergence occurs in one or the other (or both) types of topologically ordered system; or,
- (c) Deny that SRE and LRE are the appropriate mechanisms that underlie emergence in topologically ordered systems.

An example of how these options play out is given by Lancaster and Pexton's (2015) insightful analysis of emergence in systems exhibiting the FQHE. According to these authors (2015, 343), "...the presence of topological order in the FQHE is indicative of an intrinsic holism to the FQH system...". This holism should be understood as a failure of mereological supervenience that is "...located in the long-range entanglements that characterize topological states of matter" (2015, 353).

One initial concern with this proposal stems from its suggestion that long-range entanglement underwrites holism. Earman (2015, 305) observes that, in general, whether or not a state is entangled depends on the decomposition of its algebra of observables into subsystem algebras. Thus whether or not a physical system exhibits holism should be

addressed by focusing on its algebra of observables, as opposed to its state. In particular, according to Earman, holism should be associated with a failure of additivity of a system's algebra of observables, since "...this is a precise way of capturing in algebraic terms the idea that the whole is not greater than the sum of its parts" (2015, 335).<sup>27</sup>

This problem of ambiguity of entanglement can be addressed by arguing for the physicality of a particular subsystem decomposition over the others. In the case of FQHE systems, one might argue that the effect is due to electron-electron interactions, and this privileges the decomposition of an FQHE state in the single-particle electron basis. Colloquially, one might claim that the emergent FQHE system arises when electron states become long-range entangled with each other: "In the FQH state there are a vast number of electrons which become holistically tied together by long-range entanglements" (Lancaster and Pexton 2015, 354). A complication with this is that an FQHE system can be understood as a many-body interacting system of electrons, or as a one-body non-interacting system of composite fermions or composite bosons; and, importantly, the observational evidence for the effect underdetermines these contrasting theoretical descriptions.<sup>28</sup> Moreover,

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<sup>27</sup> Let  $\mathfrak{R}(\mathcal{O})$  be a local algebra of observables associated with spacetime region  $\mathcal{O}$ . Additivity requires that  $\mathfrak{R}(\mathcal{O})$  be generated by the local algebras associated with any of  $\mathcal{O}$ 's open coverings:  $\mathfrak{R}(\mathcal{O}) = \vee_i \mathfrak{R}(\mathcal{O}_i)$ , for  $\mathcal{O} = \bigcup_i \mathcal{O}_i$ .

<sup>28</sup> See, *e.g.*, Ezawa (2008, 227–8) for a discussion of the composite particle descriptions of the FQHE. The composite fermion account involves the attachment of an even number of Chern–Simons fluxes to the electrons of the fundamental system, and this serves to preserve their statistics while reducing the external magnetic field to values associated with the IQHE: in this account, the FQHE of interacting electrons is theoretically equivalent to the IQHE of non-interacting composite fermions. In the composite boson account, an odd

Shi (2004, 6814–16) demonstrates that a notion of “interaction-induced entanglement” can be defined in terms of the decomposition of a many-particle state in the basis of eigenstates of a corresponding single-particle Hamiltonian, and this yields different results when applied to the FQHE, depending on what single-particle Hamiltonian is adopted: an FQHE state is interaction-induced entangled with respect to the single-particle electron basis, whereas it is not interaction-induced entangled in either the composite fermion or the composite boson single-particle bases. Thus if long-range entanglement is explicable as Shi’s interaction-induced entanglement, then whether or not an FQHE state exhibits long-range entanglement is underdetermined by the observational evidence.

Setting aside the problem of ambiguity of entanglement, the larger concern with Lancaster and Pexton’s analysis just is, if LRE states exhibit holism because they cannot be disentangled by local unitary evolutions, then SRE states cannot be said to exhibit holism. Hence if intrinsic topologically ordered systems, like the FQHE, exhibit emergence due to entanglement-induced holism, then SPT ordered systems, like the IQHE, do not. This is Option (b) above, and the difficulty here is that, as Section 4 argued, to the extent that both intrinsic topologically ordered systems and SPT ordered systems possess those characteristics of Landau–Ginsburg ordered systems that motivate the ascription of emergence to the latter, *both* of the former should be ascribed emergence, too. In principle, of course, one might attempt to refine these characteristics, or replace them with others, so that intrinsic topologically ordered systems and Landau–Ginsburg ordered systems share the relevant traits that

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number of Chern–Simons fluxes is attached to electrons, resulting in a change of statistics and a cancellation of the external magnetic field, and the latter allows the bosons to condense at low temperatures: in this account, the FQHE of interacting electrons is theoretically equivalent to a Bose–Einstein condensate of non-interacting composite bosons.

characterize emergence, while SPT ordered systems do not; but the onus is on the mechanism-centric advocate to articulate such traits.

Alternatively, a mechanism-centric advocate could adopt Option (a); in this context, one would attempt to find an ontological underpinning for emergence, other than holism, that is applicable to both LRE states and SRE states; or different ontological underpinnings, one for LRE states and the other for SRE states, that are compatible with each other. Again, the onus is on the mechanism-centric advocate to provide these interpretations. Finally, a mechanism-centric advocate could adopt Option (c) and identify a mechanism, other than LRE and SRE, that is common to both intrinsic topologically ordered systems and SPT ordered systems; but what this mechanism is needs to be worked out.

My point is not to say these options for a mechanism-centric view of emergence in topologically ordered systems are impossible to flesh out. Rather, the point is just that a law-centric view of emergence seems to fair better. A law-centric advocate points out that both an intrinsic topologically ordered system and an SPT ordered system can be described by an effective topological quantum field theory (one example is equation 10) in the vicinity of a critical point, and by a non-relativistic local quantum field theory (*e.g.*, equation 11) away from the critical point. The fact that these theories are formally distinct, not just in terms of the forms of their equations of motion (as encoded in formally distinct Lagrangian densities), but also in terms of the dynamical variables they use to encode the degrees of freedom of the system, suggests to a law-centric advocate that the system at a critical point is dynamically independent of, and dynamically robust with respect to, the system away from the critical point. And this suggests that the system at a critical point is emergent with respect to the system away from the critical point. No further appeal to a causal/mechanical description of the system need be made.



## 6. Conclusion

If one claims that Landau–Ginsburg ordered systems exhibit emergence, then one should also claim that topologically ordered systems exhibit emergence, *both* intrinsic topologically ordered systems *and* symmetry-protected topologically ordered systems. Moreover, no appeal to a causal/mechanical mechanism is required to support these claims. One need only note that Landau–Ginsburg ordered systems and topologically ordered systems are both characterized by effective field theories near their critical points, and these effective theories are distinct in relevant ways from the theories that describe the systems away from their critical points. These relevant ways guarantee that the system near the critical point is sufficiently novel from the system away from the critical point to justify describing the former as emergent with respect to the latter.

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