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SURMOUNTING OBSTACLES: CIRCULATION AND ADOPTION OF ALGEBRAIC SYMBOLISM

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ABSTRACT

This introductory paper provides an overview of four contributions on the epistemological functions of mathematical symbolism as it emerged in Arabic and European treatises on algebra. The evolution towards symbolic algebra was a long and difficult process in which many obstacles had to be overcome. Three of these obstacles, related to the circulation and adoption of symbolism, are highlighted in this special volume: 1) the transition of material practices of algebraic calculation to discursive practices and text production, 2) the transition from manuscript production to printed works involving material limitations of typefaces, and 3) the transition of algebraic symbolism from a system of notation and representation to a tool of mathematical analysis. This paper will conclude with the observation that the whole development towards symbolization can be considered an obstacle by itself in the sense of 'epistemic obstacles' as used by Brousseau.

1. Introduction

Scholars in the history of mathematics find it difficult to pinpoint the emergence of symbolism on a time scale or even to devise a periodization for the developments leading to the symbolization of mathematics. It is meaningful that the four papers in this special issue on the subject span the very broad period from Ibn al-Yāsamīn, active in the twelfth century, over Hérigone and Mengoli publishing in the seventeenth century (Massa Esteve), to eighteenth-century books and manuscripts in Arabic (Oaks). Not only is it difficult to narrow down the history of this development to a tight timeline, the narrative can also not be restricted to a specific geographical region or culture. The papers here deal with developments in Arabic countries, the Maghreb region, the Iberian Peninsula, France, Germany and Italy, while others have pointed out ancient Greek sources as equally relevant (Nesselmann 1842, Netz 2012). To understand the process towards the acceptance of symbolism we thus have to account for multiple actors, practices, influences, cultures and locations. Clearly, the process towards symbolization was not a smooth one and many obstacles had to be overcome. In this paper we discuss some of the major obstacles that become apparent from the four case studies presented in this issue.

2. From material practices to discursive representations

Oaks spends much attention to material practices connected with Arabic arithmetic and algebra in the Maghreb region. The *takht* or dust-board allowed the representation of Hindu-Arabic numerals and facilitated calculations with such numbers. The *lawḥa* was a clay tablet providing similar functions using an inked cane stick. Evidence of calculating practices with these devices are recorded in several Arabic works.[1] In a study on the Jerba manuscript, Abdeljahoud (2002) proposed the thesis that early Arabic symbolism developed from such

[1] Oaks refers to al-Uqlīdisī and Al-Ḥaṣṣār. Abdeljahoud (2002, 17) quotes from *Kitāb al-kāmil fi sināᶜ at al-ʻaᶜ dād* by Al-Ḥaṣṣār: "In our country, calculators, especially artisans and scribes have taken the habit of using the figures they agreed among themselves to express their numbers and to differentiate them from each other (…) There are, for them, two kinds: the first is called *ghubār* or also Hindi. They gave it this name because they originally used a wooden board (*lawḥa*) on which they spread fine sand. A student in calculation uses a small stick in the shape of a pen which is used to draw figures in the sand and he calculates what he wants. Upon completion of the calculation, he wipes the sand and surface. The effectiveness [of this method] is [to enable] performing and facilitating calculations without having to use ink all the time, and to erase the board, they used sand instead of ink, and found that it facilitated calculations …" (my translation from the French).

material practices. In a survey on symbolism in the abbaco period (1300 – 1500), Jens Høyrup (2010) pointed out that the fraction bar - which is not easily associated with algebraic symbolism - developed in this Maghreb context. In the abbaco tradition the fraction bar became adopted within algebraic practice and developed into – what he coins as – "formal fractions" from the early 14th century. The Maghreb use of material aids for performing calculations was extended to algebraic problem solving already from the 12th century onwards. However, as shown in the overview by Oaks, Arabic works on algebra did not employ any symbolism and relied on a specific rhetorical style for presenting problems and solutions. It is only from the late 12th century, and confined to the Maghreb region, that symbolic representations from the dust-boards came to trickle into written works, starting with Ibn al-Yāsamīn. Only a dozen of extant works are known to adopt symbolic elements from dust-board calculations. As observed by Oaks, in the Maghreb region there were "two modes of writing algebra, the rhetorical and the symbolic", and they "served two different purposes. One would work out a problem alone using the notation, and, if needed, a rhetorical version would be composed to communicate the results to other people in a book". We thus discern here a rather strict separation between a material practice of solving algebraic problems on a dust-board and a scholarly practice of producing discursive representation of algebraic methods and problem solutions. The literary product, in the form of a textbook on algebra, was not, and was not intended to be, a representation of calculation practices that lead to the algebraic solution of problems. Such strict separation is quite remarkable and is one of the obstacles to overcome for this emerging symbolism to become accepted and disseminated.

We find a similar obstacle towards symbolism to be present in the abbaco tradition. As observed elsewhere (Heeffer 2012), abbaco manuscripts often contain non-discursive elements that function as justifications for operations and problem solutions in the explanations of the discursive text. These non-discursive elements include graphical schemes of operations (e.g. cross-wise multiplication of binomials, operations on fractions), synoptic tables summarizing the main values of a problem, scratchpad calculations (arithmetical or algebraic) and symbolic representations of polynomials and equations. Non-discursive elements are either noted down in the margin or appear as boxed elements within the text. The first conscious and deliberate use of symbolism in a family of fourteenth-century abbaco manuscripts also presents symbolic solutions as boxed elements (Heeffer 2012b). This practice of including elements *hors de texte* is further noted by Charles Burnett (2006, 29) in Latin algorisms and quoted by Oaks in relation to the Arabic treatises on Hindu-Arabic numerals.

We believe that these non-discursive elements functioned in the same way as the calculations on the *takht* or *lawḥa*. They are representations of tangible activities not directly related to the discursive text but adding to the justification and understanding of the operations explained in the text. The schemes used for operations on fractions may reflect demonstrative practices in abbaco schools (see

Figure 1), though we have no direct evidence for this. However, since the introductory chapters on fractions in treatises such as the Jacopo da Firenze's *Tractatus Algorismi*, correspond well with one of the seven *mute* known to be taught in abbaco schools (Arrighi 1965-7) we are safe to assume that these schemes served as didactical tools.

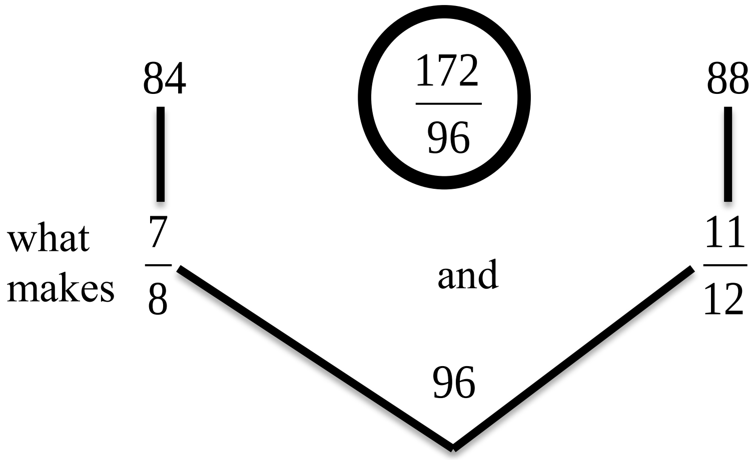

Figure 1: non-discursive scheme for adding fractions in abbaco a text (from Jacopo da Firenze's treatise of 1307, Hoyrup 2007)

The scheme suggests gestures and movements that may well have been performed in an abbaco class to teach students how to perform the calculations, in the same way that symbolic representations of algebraic entities in the works of Ibn al-Yāsamīn refer to their use on the *takht*.

In the abbaco tradition, the use of material aids for calculations was replaced by pen and paper. However, the distinction between scratchpad calculations and rhetorical explanations for the solution of algebraic problems was as strict as in the Maghreb context. In general, scratchpad solutions to algebraic problems was not considered to be suitable for use in written treatises. The family of manuscripts discussed in Heeffer (2012) are a notable exception in that the symbolic rendering of solutions is presented as an alternative to the discursive text, thus serving an epistemic function. As stated by the anonymous abbaco master who authored the treatise: “I showed this symbolically as you can understand from the above, not to make things harder but rather for you to understand it better”. However, from the fifteenth century such symbolic calculations become more and more prominent in abbaco treatises (see Høyrup 2010 for an overview). In his Perugia manuscript, completed in 1478, Luca Pacioli includes marginal calculations for most of the more intricate algebraic problems. Figure 2 shows a problem in which 100 is divided by some unknown number (Calzoni and Cavazzoni, 2007, chapter 12, problem 55, p. 408). The solution is presented in the text in the common rhetorical way. The discursive solutions do not refer in any way to the marginal scratchpad calculations.

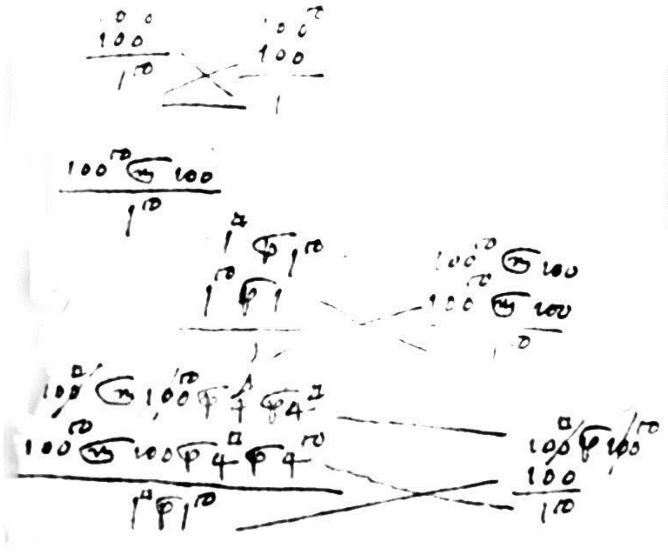

Figure 2: marginal scratchpad calculations from Luca Pacioli's Perugia manuscript

The further dissemination of such symbolic calculation practice was stalled in several ways. Firstly, the reasoning in scratchpad calculations was not understood by all scribes or by intended readers. For Høyrup (2010) the whole process was unintended and not understood by its participants. Several competing systems were in use during the fifteenth century and they were often internally and mutually inconsistent. Secondly, as it was not considered appropriate for inclusion in proper treatises. Professional scribes often expanded the symbols to words or omitted such symbolic calculations all together (see Heeffer 2012b for an example of each of these two). Thirdly, as we shall in the next session, its acceptance was further impeded by the transition of manuscript to print.

3. From manuscript to print

The first book to have one or more chapters on algebra was Luca Pacioli's *Summa de Arithmetica, Geometria, Proportioni et Proportionalita*, printed and published in Venice in 1494. As paper was responsible for half the price of the book, it was set it a very dense gothic typeface, though it still covered 615 pages (Sangster 2008). In France, the first book dealing with algebra was *L'arismethique novellement composee par maistre Estienne de La Roche* printed in Lyon in 1520 and again in 1538. The first book (almost) completely devoted to algebra was Christoff Rudolff's *Behend vnnd Hubsch Rechnung durch die kunstreichen regeln Algebre so gemeincklich die Coss genennt werden*, printed at Strasburg in 1525. It is interesting to see how the early symbolism adopted in manuscripts by abbaco masters and cossists became represented in the first algebra books. In the case of Pacioli, we have the Perugia manuscript to compare with. As shown in Heeffer (2010c), Pacioli's consistent use of algebraic symbolism in his own autograph of the Perugia manuscript of 1478 was not preserved in the *Summa*. The symbols for the unknowns is his autograph copy - which were probably the same as the Perugia manuscript - appear in the book expanded to full words. We observed the same fate with Regiomontanus's *De triangulis omnimodis*, composed in 1464 (Moscou MS. 541) but only published in 1533 by Petreus, where all of the symbolic scratchpad calculations from the original manuscript were omitted. The symbolic calculations were either considered to be unfit for publication or posed insurmountable challenges for the typesetter.

The typography of the German books compared better with the cossic symbolism of fifteenth-century manuscripts, as shown in an overview by Tropfke (see Figure 3). In the case of France, we know that de la Roche relied mostly on the manuscript of the *Triparty* by Chuquet. Some of the symbolism by Chuquet, such as the use of 'ρ' for the unknown survives in his book, but De la Roche consciously changes Chuquet's use of 1^2 for both the square of the unknown and the second unknown to avoid confusion (Heeffer 2012). He uses the ligature .1.Qtite. (1 quantité) for the second unknown and the German cossic symbol for the second power of the unknown.

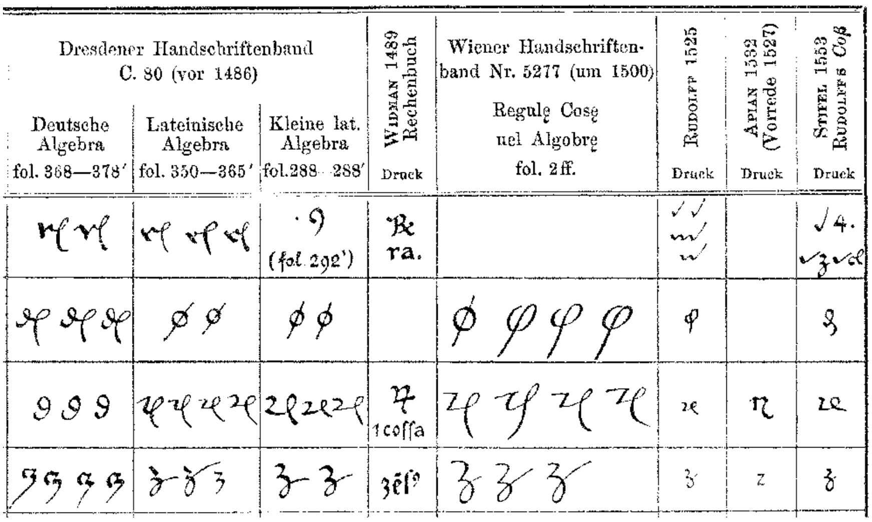

Figure 3: symbols for the root, number, the unknown and the square of the unknown, from manuscript to print (from Tropfke,)

François Loget (2012) looked at the role that printers had in the further spread of algebraic symbolism, in particular Jean de Tournes, Chrétien Wechel and Guillaume Cavellat. Together they were responsible for some of the most representative algebra books printed in France in the middle of the sixteenth century. For Loget, the major obstacles for the transition of symbolism from manuscript to print were the reluctance of printers to use the new spelling and the cost of cutting new font. From a study by Nina Catach (1968) cited by Loget, it appears that only from the middle of sixteenth century the authors exerted an influence on the printers and force their view on typographical innovations as were needed for algebra books. Loget points at Petrus Ramus and Jacques Peletier as two examples of authors that had an auspicious relationship with their printer and could influence the publication. Earlier, decisions were mostly made by the printer's craftsmen. Only the more prosperous printers were able to cut new type to oblige with new authors demands.

Not mentioned by Loget, but relevant for the dissemination of symbolism is the idea of typographical fixity.[2] Elisabeth Eisenstein (1979) coined the term in her influential work on the impact of the invention of printing in Europe. Her idea of fixity was heavily criticized in a similar work by Adrian Johns (1998). However, we believe that the idea of typographical fixity has some merit for our discussion. Firstly, we have to remark that Eisenstein uses the term in three different meanings, each one describing a different effect of the printing press:

- An effect of printing which gives words and ideas a permanence (p. 118, 119, 120, 181, 182, 311, 326)
- Preservation of signs and symbolic representations (ancient languages, maps, charts, symbols) (p. 113, 116)
- Construction of new representations and visual aids by recycling and rearranging existing ones in printed form (p. 255, 270)

In Johns criticism 'typographical fixity' is used in the first sense only. Obviously, with regards to symbolism the two other interpretations have a direct application to all algebraic works following Pacioli's *Summa*. Particularly the cossic symbols for the powers of the unknown

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[2] I have raised this topic before in relation to algebraic symbolism in Heeffer 2006.

from Rudolff (1525) became pervasive for the next centuries, despite their shortcomings. The case study by Fàtima Romera Vallhonesta (2012) shows how they became fixed in publications on the Iberian Peninsula. Marco Aurel, from German descent, introduced the cossic symbolism in Spain and it dominated further publication on algebra during the sixteenth century. Michael Stifel improved on the cossic symbolism by introducing letters *A*, *B*, and *C* for unknowns in linear problems, in his influential *Arithmetica Integra* of 1545, and *AA* for the second power of the second unknown in his edition of Rudolff's *Coss* of 1553 (Heeffer 2010). Stifel's *Arithmetica Integra* was the basis for the *Algebra* by Clavius (1608) and this work fixed the symbolism of all Jesuit works of the seventeenth century, such as *Abrégé des préceptes d'algèbre* by Jacques de Billy from 1637. As shown by the research by Catherine Jami (2007) Jesuits such as Antoine Thomas (1644-1709) still embraced the fifteenth-century cossic symbolism when introducing Western algebra to the Kangxi emperor around 1700. The emperor was not impressed.

Johns (1998) has argued against Eisenstein that typographical fixity occurred only in rare circumstances and when it did happen, it was usually many years after the invention of printing. I endorse his view that typographical fixity in the first sense is not an intrinsic quality of printing but a transitive one. However, I do feel that for symbolism in algebraic textbooks it became an intrinsic quality. Once symbolic solutions became commonly accepted in print their image became ubiquitous and subsequently readers found it difficult to understand problem solution presented in the old rhetorical way. Typographical fixity became a crucial factor in the acceptance of symbolic algebra after Viète and Descartes.

4. From symbolic notations to epistemic tools

In her study on the symbolism of Viète, Hérigone and Mengoli, Maria Rosa Massa Esteve (2012) shows how symbolism had to overcome one more obstacle in the transformation towards early-modern mathematics. Where symbols had been used as representational tools by Cardano, Stifel and the French humanists, they became epistemic tools for abstract analysis for the adherents of the new *logicistica speciosa*. This new approach transformed algebra from algebraic problem solving to the study of the structure of equations (Mahoney 1980). In order to do so, Viète had to introduce one missing component to the arsenal of symbolic instruments by the end of the sixteenth century. Michel Serfati (2010) has coined the term "the dialectic of indeterminacy" for it, one of his six patterns of symbolic representation (Serfati 2005). Viète (1591) distinguished the "given and required magnitudes" by consonants and vowels respectively. His use of the letters *A*, *E*, *I*, *O*, *U* to represent the unknowns, was not very different from what Stifel had proposed in 1545 and improved in 1553. However, his use of the abstract symbols *B*, *C*, *D*, *F*, .. *Z,* for given but undetermined quantities was completely new. This allowed him to study equations in a more general way. For example the equation $A^2 + AB = Z^2$, by Viète written as "A quad plus A in B aequalis Z quad", which after Descartes (1637) would be become $x^2 + ax = b^2$ could express a specific relation between given and unknown magnitudes. In this example the equation is derived from a problem about magnitudes in continuous proportion, which in modern symbolism can be represented as $\frac{a+x}{b} = \frac{b}{x}$. The resulting equation expresses the structure of the problem, not for some specific values as in algebraic problem solving before Viète, but for *all* problems with that structure, independent of the specific values of the problem.

Massa Esteve (2012) nicely shows how this specific aspect becomes important in the algebra of Hérigone and Mengoli. She argues that it allowed Mengoli to arrive at a complete new concept in 1659: determinable indeterminate quantities. This changed the concept of an algebraic unknown into that of a variable. The "specious" language, which in the sixteenth century had a representative function, became for Mengoli an analytic tool. It is a nice illustration of how symbolism itself facilitated the transition of symbolic language as a representational instrument to an epistemic tool. This bootstrapping process - symbolism augmenting the functions of symbolism - became crucial by the end of the seventeenth century. Leibniz, the master-builder of mathematical notations build layers upon layers of symbolism to construct new mathematical ideas and concepts (Knobloch 2010).

5. Conclusion

The contributions in this special issue on the epistemic aspects of symbolism each highlight some kind of historical obstacle that needed to be overcome: the integration of symbolism used in material practices within scholarly texts, the survival of symbolic practices in the switch from manuscripts to printed books and the transition from the use of symbolism as a notation system to the use of symbolism as an epistemic tool. However, the whole process of the emergence of symbolism in mathematics can be considered by itself as an 'epistemic obstacle', a term coined in 1938 by Bachelard within the context of history of science. The original term refers to misleading elements blocking the rational process of advancement of science. The idea was adapted by Brousseau (1976) for use in mathematics education. Brousseau attributes a positive function to epistemological obstacles within his didactical project. He considers such obstacles more as a piece of mathematical knowledge rather than a lack of knowledge. He identifies them in the history of mathematics as well as students' spontaneous models. In classroom situations they do not appear as erratic or unexpected errors, but as predictable ones. Indeed, the tortuous process towards mathematical symbolism is often reflected in students' difficulties with mastering the principles of elementary algebra. The symbolism of mathematics is often cited as the most important obstacle in learning mathematics. It also acts as a barrier for historians to grasp and understand the meaning of mathematical concepts, operations and practices before the emergence of symbolism. A historical understanding of the epistemic functions of algebraic symbolism thus provides us with insights beneficial for mathematics education as well as for the history of mathematics.

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