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# NOMINALISM AND CONVENTIONALISM IN SOCIAL CONSTRUCTIVISM

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## 1. Introduction

There are various forms of social constructivism in the social and human sciences, especially in sociology (Berger and Luckmann 1966), discursive psychology (Gergen 1999) and philosophy (Hacking 2001). There are also variants in education and learning theory (Wertsch 1991), in mathematics education (Weinberg and Gavelek 1987), and even in the philosophy of mathematics (Hersh 1997). However, what is meant by the use of the term here is that version of social constructivism as a philosophy of mathematics given its fullest expression to date in my eponymous book (Ernest 1998). To avoid cumbersome circumlocutions, no reference will be made to the range of extant or possible variations or alternatives, although it is not intended to assert the superiority or supremacy of the singular version explicated here.

Social Constructivism is put forward as a philosophy of mathematics with the primary aim of offering an account of mathematical practice, including also the social structuring and the historical development of mathematics. It is a naturalistic philosophy and so it has many elaborated characteristics relating to its descriptive and social aspects. However although ontological and epistemological issues are discussed in Ernest (1998), I wish to focus here on its relationships to traditional ontological and epistemological positions in the philosophy of mathematics. While social constructivism has been contrasted strongly with Platonism and mathematical realism, on the one hand, and with foundationalist and absolutist positions, on the other hand, here I wish to elaborate its positive location in philosophical traditions.

This paper cautiously describes social constructivism as nominalist, with respect to ontology, and conventionalist with respect to epistemology and the foundations of knowledge. The caution is due to the particular variants or interpretations of nominalism and conventionalism that are attributed to social constructivism, for there are dominant traditions within each perspective with which social constructivism is not identified or subsumed under.

By asserting that the objects of mathematics are signs, rather than purely psychological or mental entities (the claim of conceptualism), material entities (the claim of materialism) or objective self-subsistent entities (the claim of realism) what is espoused is claimed to be a variety of nominalism. However, this differs from the most common forms of nominalism, for it is based on a conception of sign that rejects the representational theory of truth. Signs are part of a cultural realm that is intersubjective. It transcends the perceptions and understanding of any one individual but does not transcend the knowledge and practices of humankind as a whole and thus does not belong to any extra-human reality. Elsewhere I have termed this realm 'objective', but this requires a new definition of the term, differing from the received, traditional usage and its ontological presuppositions (Ernest 1998).

The related claim that the concepts, terms, theorems, rules of proof and logic, truths and theories of mathematics are socially constructed cultural entities constitutes a form of conventionalism. However, what is not asserted is that the concepts and truths of mathematics are the result of arbitrary, whimsical or even ideologically motivated decisions and choices. Many conventions in mathematics are not conscious decisions but reflections of historical practices laid down for very good reasons. Furthermore, where conscious decisions are made in mathematics they are usually to complete or extend existing rules and practices within mathematics that result in general, simple, elegant, practical and consistent systems. As such, such choices although not actually forced or necessary, for if so they would not be choices within the sense meant here, are nevertheless the choices that come closest to being required by past traditions.[1]

## 2. Nominalism

Social constructivism is claimed to be a nominalist philosophy of mathematics because it asserts that the objects of mathematics are signs. I understand signs in the semiotic sense of having or being composed of both a signifier, that which represents, often a material representation, and a signified, the meaning represented. Thus unlike in Hilbert's formalism, the signs of mathematics are not just detached and empty symbols (signifiers) but always have meanings (signifieds), even if precisely specifying the characteristics and ontology of these entities is difficult and complex, and may involve ambiguity and multiplicity. Following Peirce's seminal work in semiotics, as well as modern semiotic philosophy (Eco 1984, Derrida 1978), these signifieds are understood to be further signs. Although Peirce developed a complex tripartite treatment of signs, he is unambiguous in asserting that "The meaning of a representation can be nothing but a representation." (Peirce 1931-58, Vol. 1, Section 339). In Peirce's system the 'interpretant' of a sign corresponds in many respects to the meaning or interpretation of a sign.

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[1] Wittgenstein (1953) uses a broader notion of choice in following a rule, claiming that any rule following involves an agreement or decision, in the sense that choosing to participate in a language game and form of life and maintain its rules and conventions, such as respecting Modus Ponens in mathematical proof, is always a matter of choice, not necessity. Without challenging this, the choices, conventions and decisions I am referring to are those that come about when there is no unequivocal and unambiguous already laid down rule to follow. Thus William Hamilton in inventing, constructing and defining the system of Quaternions chose to abnegate commutativity in the binary operation in his system (i.j = -j.i   j.i) to obtain the best system with the properties he sought, although other options were open to him (Pickering 1995). This was not a permitted move in the contemporary language games of algebra, and thus led to the formation of new language games that have proved very fruitful in mathematics.

> So there is an infinite regression here. Finally, the interpretant is nothing but another representation to which the torch of truth is handed along; and as representation, it has its interpretant again. (Peirce 1931-58, Vol. 1, Section 339).

Although there is potentially an infinite regression in this theory, this is no more a vicious circle than looking up meanings in a dictionary. Ultimately the dictionary meaning of a word can only be given in terms of relationships between other words and their meanings. So too, I claim, the meanings or signifieds of virtually all signs in mathematics are themselves further signs. I do not claim that every meaning is itself a sign because I want to leave open two further possibilities: first, that signifieds can be actions or operations on signs, and second, that they can be tangible objects in the real world. I do not at this stage think that the second case is relevant to mathematics, for if, for example, one points to a set of three objects as an instance of the number three, in my view this indicated plurality or set constitutes a sign itself, rather than just the meaning of a sign. Indicating a set of tangible objects in effect makes it a sign itself.

From a nominalistic perspective, it would not be possible to consistently adopt any other definition of meanings except in terms of other signs. For any other definition would swiftly lead out of a nominalist theory into something else. If meanings were in general located in the physical world, this would give rise to an empirical realist ontology. If the meanings were located in some abstract realm, the outcome would also be a realist ontology of some abstract sort, like Platonism for example. Likewise meanings located in the mind would give rise to a form of conceptualism.

Thus the ontological position of social constructivism is nominalism. The objects of mathematics are signs, and furthermore the meanings of these signs are typically yet further signs. However, as indicated above, this is not to say that all that exists in mathematics is signs. There are in addition sign related activities: idealized human action on signs. Thus the numeral '3' connotes both the act of establishing a one-to-one correspondence with prototypical triplet set (cardinality), and the act of enumerating a triplet set (ordinality). Each of these connotations presupposes some elements of threeness, requiring the use of a representative triplet set. But the explicit definition of 3 as the successor of 2 does not. Formally this definition can be represented as 3 $=_{def}$ S2, where S is the one place operation of successor, a primitive in the theory of Peano arithmetic. Although this identity asserts a static relationship, it contains an operation, namely that of applying the successor operation to 2. Traditionally mathematics is understood as timeless, since the idealized act of transforming 2 into its successor S2 (commonly named '3') is reported after the idealized act, with no temporal framework. Nevertheless, the admission and representation of change signifies an abstracted and idealized analogue or counterpart to time. Of course the idealized analogue of time admitted in mathematics differs in one major respect from time in its usual sense, it is reversible. $S^{-1}3 = 2$, just as truly as S2 = 3, where $S^{-1}$ denotes the inverse operation of S (the predecessor operation).

In learning the meaning of the equals sign (=) children usually first understand it as a sign of sequential transformation (see, e.g., Kieren 1992). So 2+3 = 5 is read as 2 added to 3 makes 5, in which the equals sign signifies the move to the end product of an operation. Consequently, initially children often have difficulty in grasping its symmetric form 5 = 2+3 since this is not easily understood in these terms.[2] A necessary step in the successful learning of mathematics is to grasp the symmetric property of equality, as well as its reflexivity and transitivity. Likewise, these three properties must also be understood to apply to the analogous relation of logical equivalence that applies to formula and sentence pairs. Thus developing an understanding of the signs of mathematics involves abstracting from the time sequential feature of operations, because of the abstract and timeless character of the signified space of mathematical operations and objects.[3] Of course the signs of mathematics are nuanced enough in their meanings so that potentially irreversible sequences of

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[2] This is related to the need for premature closure noted and defined by Collis (1975). It represents the empirically noted desire of children in the early stages of learning mathematics to achieve syntactical simplicity by deriving a single answer in working a mathematical task.

[3] Indeed Piaget (1952) had the prescience to identify the stage of being able to reverse informal mathematical and logical operations as a crucial step in the development of children's mathematical thinking. The achievement of Invariance in this sense signals the transition to the stage of Concrete Operations.

operations can be represented too, through both ordered relations on terms and formulas (using, e.g., $>$ and $\rightarrow$ , respectively) and reductions in complexity of terms and formulas (using, e.g., substitutivity of terms). The spaces of transformations of mathematical signs can have both abelian and non-abelian group-like properties (although in most case they do not strictly speaking form a group).

The discussion of this very simple example begins to illustrate some of the richness and complexity of mathematical signs and their meanings, understood semiotically and nominalistically. There are signs, sign rules, and meanings for both signs and their rules. Furthermore these meanings occur in two forms, first as agreed by the mathematical and educational communities (shared or conventional meanings), as well as learners' own developing and sometimes idiosyncratic interpretations of them.

To understand even a part of a simple mathematical topic involves mastering a complex set of inter-relationships between signs, sign rules, and sign meanings, which themselves embody complex relationships between yet further signs and rules of sign use. Thus the nominalistic claims of social constructivism are not purely theoretical, but also describe some of the dimensions of coming to know and understand in mathematics, that are presupposed if not always acknowledged by mathematicians and successful users of the subject.

Azzouni (1994) distinguishes two forms of nominalism. The first form simply denies that there are abstract mathematical objects. This form leads to a variety of problems. As Quine (1953, 1969) has argued, allowing quantifiers to range over classes of mathematical objects, such as sets or numbers, and then denying that these objects exist, is problematic. To be carried through consistently this position leads to cumbersome circumlocutions in which the objects are used but then defined away, as mere *façons de parler* in place of complex and multi-layered definitions. This is not the nominalist position of social constructivism.

The second form of nominalism does not deny that mathematical objects exist, but instead concerns "the *identification* of them with *some* of the notation supposedly referring to them" (Azzouni 1994: 47). This more or less corresponds with the position adopted here. Azzouni goes on to call this 'nominalism on the cheap' because it avoids the complex or in his metaphor, costly, circumlocutions of the first form. He makes a final telling remark which draws him close to the ontological position of social constructivism: "mathematical objects *are* posits, and posits are not, strictly speaking, independent of their positors." (Azzouni 1994: 214; original emphasis throughout).

Likewise, the nominalist position adopted here is that mathematical objects are the meanings of signs, themselves signs or actions and operations upon them. But then this raises the further problem of what signs are and in what space they are to be found. As the previous quotation suggests, the social constructivist position is that they are humanly made posits, but this of itself does not clarify their ontology. A useful starting point for classifying spaces or domains of existence is Popper's (1979) 3 worlds, the physical, mental and objective. While I do not subscribe to any absolute tripartite division such as this, it provides a useful framework for discussing signs and knowledge that captures many of the traditional and still widespread notions of ontology.

As signs, the objects of mathematics do not fall neatly into any one of Popper's 3 worlds. World 1 is the material and physical world. Signs have material representations in this domain, and cannot exist without them, but they are more than these representations. For example two utterances of the same sign each have unique material representations and yet are representations of the same sign. The relation of identity that holds between them, and *a fortiori* between any two tokens of the same type, is not materially present except as represented in a third possibly unwritten sign. At the next higher level of abstraction, two apparently identical signs in different contexts often will have different meanings. Thus there is more to a sign than its mere material representations. There are also relationships between signs manifested as human understandings and rules of sign use.

World 2 is the mental or psychological world. Signs undoubtedly give rise to personal meanings which some might wish to locate in this world.[4] But if meanings were solely located here there would be

[4] My cautious phrasing is because in social theories of mind the personal need not be identified with the psychological. See, for example, Harré and Gillett (1994) Vygotsky (1978) and Wertsch (1997).

problems with communication and agreement.[5] Wittgenstein (1953), in his Private language argument, makes the case that we do not and cannot have private languages that refer to our private sensations and experiences. Languages are primarily public, as they are deployed and developed in social language games, which are part of shared forms of life. Only after we begin to acquire mastery of language through public use and performance do we internalize and appropriate them to our personal world of meanings.

World 3 is the objective world of concepts, meanings, problems, knowledge, etc. The positing of this world as the repository of objective meanings and the objects of mathematics provides the basis for realism and Platonism. This is an elegant and self-consistent family of ontological positions that has satisfied scholars from Plato to Frege, Gödel, and Popper, as well as most mathematicians and philosophers of mathematics. Part of its appeal is that it accommodates the undeniably objective and impersonal aspects of mathematics so much better than Empiricism and Conceptualism.

The social constructivist solution is to adopt World 3 as a domain of objective knowledge, but to redefine it as social and cultural. This is to adopt, at least in part, the social theory of objectivity proposed by Bloor (1984), Harding (1986), Fuller (1988) and others.

> What I mean by saying that objectivity is social is that the impersonal and stable character that attaches to some of our beliefs, and the sense of reality that attaches to their reference, derives from these beliefs being social institutions. (Bloor 1984: 229).

Bloor argues that Popper's world 3 can defensibly and fruitfully be identified with the social world. He also argues that not only is the three-fold structure of Popper's ontology preserved under this transformation, but so are the connections between the three worlds. However, this social interpretation does not preserve Popper's *meaning* for objectivity, although it accounts for most features of objectivity: the autonomy of

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[5] This is a key problem for versions of Intuitionism that claim that the objects of mathematics are personal concepts, but that different person making individual acts of construction arrive at identical concepts.

objective knowledge, its external thing-like character, and its independence from any knowing subject's subjective knowledge.

Applying this to the objects of mathematics, the claim is that these objects are signs and their operations. Signs and sign operations have material representations (World 1 manifestation), they give rise to and evoke meanings in people (World 2 manifestation), but they are social institutions (World 3 manifestation). Signs and their operations are social institutions because they rest on shared functions, rules and agreements and these govern their uses and shared meanings. Sign usage and meanings are both learned and validated in public, in Wittgensteinian terms, in language games situated in social forms of life. However, the public rules and agreements underpinning the use of signs are not all explicit. Some of them are tacit, embedded in custom and accepted practices, and acquired implicitly through social participation in language games. A good analogy, although it is a special case, is the learning of the rules of grammar through participation in linguistic practice. Many grammatical speakers cannot explicitly state the rules of grammar. Rather they have induced them, as if by osmosis, from the patterns of accepted (and criticized) language use they have experienced as speakers and listeners. Consequently, they have learned them implicitly as patterns that guide the production and reception of speech, and similarly, of written expressions.

Unlike written alphabetic language, mathematical signs are not simply constructed to express or tell a story, with superfluities and decorations and the possibility of many substitutions of words or the addition of phrases or sentences. Mathematical signs are closely worked in transformational sequences to express operations on signs, be they primarily calculations or proofs. To fulfill these roles, satisfy the underlying rules, and correctly express the compound operations involved, there are very tight constraints on the selection and uses of signs. Each sign must be justifiably linked through rule applications to its predecessors and its successors in the sequence of signs that constitutes a calculation or proof. Such rule applications ensure that there is a continuity and preservation of meaning, such as numerical value or truth value, throughout the length of the sign sequence. Above all, a sequence of mathematical signs is goal directed, it is intended to end with a calculated value or proved theorem.

The meanings underlying mathematical signs, themselves signs, are the objects of mathematics, in the case of terms, or statements of relationships between the objects of mathematics, in the case of formulas and sentences. There is no other realm of objects to which the language of mathematics signs points. The analogy with alphabetic language fails, for the latter points to the lived world of human beings, as well as their imagined worlds, although it too can point to the world of signs.[6]

Like mathematics itself, signs are objective, they transcend the individual and are owned and created by humankind. Individuals may partake of either, and can add a little bit to the uses of either that may, if adopted more generally, be incorporated into the larger pattern owned by humankind. This parallel, I claim is not a coincidence. The worlds of mathematics and signs are so similar because they are the same in certain respects, in that the objects of mathematics are signs. Of course the converse does not hold, because there are non-mathematical signs such as street signs, medical symptoms, paintings and novels.

The relationship between human beings and the cultural and social worlds of signs and, *a fortiori*, of mathematics is not a symmetric one. Focusing on mathematics alone, it can uncontroversially be stated that mathematics is much larger than any one person or group of persons. For although individuals and groups can make even large contributions that become accepted into mathematics by common consent, such contributions are always tiny when viewed against the notional totality of the discipline that has evolved over the past 5000 years. In learning mathematics, that is appropriating some of its sign and meaning systems, no-one can take on board the totality. In making mathematics, the novel sign uses, patterns and meanings will always be a small addition in comparison to that which was appropriated. The enduring vastness of the discipline gives it its objectivity and external thing-like character.

One of the dangers of Popper's (1979) tripartite ontology is that it splits human action into three separated realms, and this inevitably has an impact on the interpretation of signs. Clearly human beings have a physical and material basis, and their mental activities, I would argue, are inseparable from this basis. But they are also social and hence are both

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[6] Phonetic language also points to the world of human vocal sounds, but this is irrelevant here.

constituted by their relationship with signs, culture and knowledge, as well as creating these inhabitants of World 3. Furthermore, the relationship between these three worlds is not static, but a dynamic and interpenetrating functional dance or conversation. That is, they all run together seamlessly in the world, including all activities, such as, for example, me the writer writing this text and you the reader reading it.

Signs are thus not static objects, but functions and tools that are continuously at work, used by human beings in their communicative and other practices. Thus when I claim that mathematical objects are operations on signs and sign functions as well as the signs themselves, this is not an optional add on. Signs only have meaning within the nexus of human sign related activities.

In his analysis of the semiotics of mathematics Rotman (1994) distinguishes several categories of mathematical signs. First he distinguishes the alphabetic from the numeric (or mathematical) sign. Numeric signs are further analysed into the ideogrammatic and the diagrammatic. Ideograms include numerals, =, +, and so on, signs that can be inserted within the one dimensional flow of alphabetic signs. Diagrammatic signs are complex, relational, and typically cannot be inserted in the one dimensional flow of alphabetic signs, as they are multi dimensional. To use such signs in mathematical texts requires a break in the alphabetic flow. Although the origin of such breaks lies in ancient mathematical and scientific texts, we now take for granted the insertion of tables, figures, diagrams, etc., in texts of all sorts. Indeed, in hypertexts even videos and links to other texts are inserted in this way.

Rotman does not use the dimensional distinction applied here. However, Lemke (2003) distinguishes typological versus topological semiosis in mathematics, which loosely corresponds to the alphanumeric versus diagrammatic distinction in Rotman, and invites the dimensional distinction.

Rotman, Lemke and others argue for the necessity and irreducibility of the diagrammatic and topological modes of semiosis in mathematics. This provides an argument against Logicism. For if mathematics could be reduced to logic, then the diagrammatic would be dispensable in mathematics. For despite Frege's (1879) two dimensional Concept Writing, mathematical logic is expressible alphanumerically, that is in terms of alphabetic and ideogrammatic symbols. Mathematical logic could be expanded to accommodate the diagrammatic signs of mathematics, but this contradicts the claims of Logicism. Namely, that the concepts of mathematics can be defined in purely logical terms and the theorems of mathematics can be derived solely from logical axioms. But diagrammatic signs are not a part of modern mathematical logic. If the diagrammatic and topological modes are ineliminable in mathematics, then mathematics cannot be reduced to logic. Logicism has already failed to establish its second claim in that the theorems of mathematics cannot be derived from purely logical axioms. However, this argument means that Logicism has also failed to establish its first claim in that the concepts and signs of mathematics cannot be defined in purely logical terms.[7]

Overall, what I am claiming is that the objects of mathematics are to be found in the domain of mathematical signs. Mathematical objects as are not only named by signs but are also brought into being through sign functioning. Each mathematical theory as it is defined opens up a domain of mathematical discourse, and this domain is populated by mathematical signs, the objects of mathematics. On a metalinguistic level these theories themselves are mathematical objects too. Within mathematical theories definitions bring new objects of mathematics into being, and the permitted modes of definition are strictly regulated. For example, in Peano arithmetic '+' is defined inductively, that is by means of the induction axiom. Typically, this definition is as follows: '+' is a binary operation defined on the set of natural numbers, so that for all natural numbers x, $x+0 = x$ and $x+Sn = S(x+n)$. This definition creates the addition operation on N, and although it is not explicitly defined so that it can be eliminated in terms of its definiens in a single linguistic

[7] I have to be careful what I assert here, because the diagrammatic and topological modes are representable in terms of the alphanumeric and the typological, just as pictures can be digitized. This follows from Descartes' groundbreaking linking of algebra and geometry. However in making such translations and reductions a human faculty of knowing is eliminated. We have both spatio-visual and logico-linguistic modes of knowing (often identified with right and left brain hemisphere activities), and the elimination of one in favour of the other loses some of the balance, complementarity and power of our thought. The fact that all knowledge can be represented in binary code does not mean that human knowing is enhanced by actually representing all knowledge this way. Such reductions threaten or even destroy meaning and understandability.

move, the definition is primitive recursive and hence constructive. That is, by iterative substitutions x+n is equivalently transformed into SS…Sx, the successor of x repeated n times. At each stage the substitution is explicitly defined, and there is an initially given limit to the number of substitutions required.

What even this very simple example illustrates is how mathematical objects are mathematical processes or their end products. In the words of Machover (1983) mathematical objects are reified constructions, that is they are the reification or nominalisation of constructive procedures and processes. That is the operations on mathematical objects themselves become mathematical objects but at the next higher level of abstraction or definition. This is reflected in the mathematically primordial transition from ordinal counting and numbers to cardinal numbers, but it is also reflected at every level in mathematics as new objects are defined in terms of old. Not every definition involves a reification, unless one counts the collecting together objects to form a finite set. (Evidently forming an infinite set does involve a reification). However, unlike Machover I do not limit the domain of meaningful constructions to those acceptable to constructivists or Intuitionists.

From a semiotic perspective, forming a completed infinite set is no more objectionable in principle than forming a finite one, even though the constructive processes involved cannot be completed before the reification into a new object. Rotman (1993) in his groundbreaking semiotic theory and philosophy of mathematics has qualms about admitting unfinishable imaginary actions on mathematical signs. But my view is that in the virtual reality of mathematical signs and objects, provided our sign practices are consistent and conservative of the underlying meanings, no types of actions on signs need be ruled out *ab initio*. Evidently part of the strength of mathematical sign systems is that they can incorporate infinities and unfinishable procedures and actions, because they are purely semiotic systems. They are not constrained by the finiteness of matter, energy limitations and entropy we accept of the physical world.

It might be objected that the semiotic interpretation of mathematical objects as signs that I have sketched is not a form of nominalism, but a version of mathematical realism or some other doctrine. For I do not reject or deny the existence of abstract mathematical objects, but locate them in the cultural and discursive domain of semiotics. But traditional philosophy typically 'looks through' language to find the domain of objects to which it refers, analogous to the correspondence theory of truth or the picture theory of meaning. I am asserting that there is no other reality in which to find the objects of mathematics other than the domain of mathematical signs and their uses and functions in the domain of discourse. This certainly preserves many of the properties of mathematical realism and Platonism, just as the social interpretation of objectivity preserves properties of Popper's World 3. Since nominalism is the doctrine that the objects of mathematics are just names, or more generally signs, this seems closest to the spirit of a semiotic, social constructivist philosophy of mathematics. However, if nominalism is understood to mean a position that denies the existence of abstract mathematical objects such as numbers and sets in any respect, this is not the position adopted here.

Putnam (1972) characterizes nominalistic language as formalized languages whose variables range only over individual things, and whose predicates stand only for adjectives and verbs applied to individual things. Such languages cannot make reference to numbers, sets or other abstract objects in any respect. He criticizes such language as inadequate for either mathematics or science. By implication, this is a critique of any strong version of nominalism that restricts itself to nominalistic languages. This is clearly not the version of nominalism I am endorsing here, which is supposed to support and recognise existing mathematical practice, which clearly involves the abstract objects of mathematics, rather than restricting it or requiring its reform. The nominalism adopted here is supposed to address the issue of what these objects are, and where they are to be found, rather than to try to define them away.

Burgess (1983) argues against all forms of nominalism, which he characterizes into three types: instrumentalist, hermeneutic and revolutionary nominalism. Each of these relates to the role of abstract mathematical objects in science. Instrumentalist nominalists adopt an instrumental philosophy of science, hermeneutic nominalists argue that on linguistic analysis the need for abstract mathematical objects like numbers is eliminable, and revolutionary nominalists claim that a new kind of science is possible in which no existential claims about abstract mathematical objects are made. Burgess criticizes and rejects each of these three positions, beginning with instrumentalism. This ends up, he argues, through rejecting the truth of scientific theories leading to the rejection of common sense beliefs as well. In addition, he doubts whether it is possible to truly disbelieve scientific theories that make reference to mathematical objects. But the position I wish to support here is precisely the instrumentalist one he rejects, although by no means expressed in his terms. Social constructivism is instrumentalist about mathematics, science, language, and all of human cultural creations. To assert this requires me to anticipate the conventionalism I endorse in the next section. Namely, that our intellectual creations are shared and jointly created conceptual tools for understanding and operating in the physical, social and cultural worlds we inhabit.

My interpretation of Wittgenstein supports the semiotic versions of nominalism espoused here. For as Wittgenstein asks:

> Is it already mathematical alchemy, that mathematical propositions are regarded as statements about mathematical objects - and mathematics as the exploration of these objects? In a certain sense it is not possible to appeal to the meaning of the signs in mathematics, just because it is only mathematics that give them their meaning. (Wittgenstein 1978: 99)

Wittgenstein suggests that we need to look within mathematics itself for the meaning of the signs of mathematics, and to project them outward into some realist or Platonist domain of mathematical objects is a delusion brought about by our linguistic habits.

Although the version of nominalism I am espousing here has not received much attention in either the philosophy of mathematics or in philosophy in general, it offers a number of explanatory benefits. First of all, it opens the door to an evolutionary epistemology and genetic epistemology approach for understanding mathematics. Mathematical signs, terms, concepts, and theories have grown more complex and elaborate over the course of 5000 years of recorded history, and so too has the range of abstract entities signified by mathematics. Understanding mathematical objects nominalistically in terms of signs and sign use explains how the realm of abstract objects of mathematics can have grown with the course of history. If the objects of mathematics are already pre-existent in an objective realm it seems strange that we perceive them and relate to them only when we are able to construct them.

Secondly, persons from school children to adult mathematicians have access to a varying and developmentally growing range of signs, rules and meanings. The mastery of signs and their rules and meanings is central to all communicative activity, so it is not surprising that it is central to mathematical understanding too. To view mathematical objects as existing in some objective realm beyond the grasp of our actions just seems less plausible than locating it in our communicative practices.

These are not very persuasive arguments to someone who rejects nominalism as described here. But they do represent benefits to someone open to the possibility of this means of accounting for mathematical objects in that they allow the resultant philosophy of mathematics to fit together with historical and developmental (psychological) accounts of mathematics to give a coherent overall view of the field.

## 3. Conventionalism

Conventionalism has been described as "the view that a priori truths, logical axioms, or scientific laws have no absolute validity but are disguised conventions representing one of a number of possible alternatives". (Norton 1997: 121)

This captures one of the key claims of social constructivism, namely that the concepts, axioms, truths, theorems, theories and standards of mathematics have no absolute validity but represent one set of choices or possibilities out of a number of possible or imaginable alternatives. This is not to critique or denigrate the excellently fruitful and valuable choices that we inherit from history, or elaborate in contemporary mathematics. Far from it, the choices and their outcome, the discipline of mathematics, represent one of the pinnacles of creative flowering of the human spirit. As free creations of humanity, rather than something pre-existing forced upon us by inevitable necessity, the ideas and results of mathematics are all the more remarkable for their beauty and elegance, yet powerful generality and practical utility.[8] Thus, the

[8] The creations of mathematics are free in a strictly regulated and restricted sense that except in exceptional cases conserves existing meanings, rules and structures.

claim that I am making is that mathematics rests on human conventions, choices and historical practices in a way that I shall elaborate.

There is a conventionalist tradition in the philosophy of mathematics which acknowledges conventions, fundamentally social agreements of one sort or another, as providing the basis for logical and mathematical proof and truth. Elements of a conventionalist philosophy of mathematics are to be found in Poincaré (1905), who asserts that certain geometrical hypotheses are freely but not arbitrarily adopted conventions. Similar elements are also to be found in Ajdukiewicz, who termed himself a radical conventionalist, and argued that the linguistic basis of knowledge significantly determines its content (Giedymin 1982). Logical empiricists such as Carnap, Hempel, Nagel, and philosophers such as Ayer, Quine, as well as others, espouse versions of conventionalism.

There are different forms of conventionalism, which hinge on different interpretations of the concept of convention. Fuller (1988) makes a key distinction between two senses of the term. First, there is the more artificial sense of convention as an explicit agreement on a definition, assumption or a rule. This might be termed a *rational convention*. The standard versions of conventionalism in the philosophy of mathematics referred above adopt this sense and propose in one way or another that the conventions on which mathematics rests, i.e., the foundations of mathematics, are chosen for pragmatic reasons. This is closely related to instrumentalism, in which theories are tools chosen to serve specific purposes.

According to what might be termed rational conventionalism, the body of mathematical knowledge, the superstructure, follows by logical means, i.e., proofs, from its conventional basis. In the extreme form which Dummett (1959) termed 'full-blooded' conventionalism and incorrectly attributes to Wittgenstein, there is no base-superstructure division and all mathematical knowledge is directly adopted by convention. This latter position is untenable, given the universally acknowledged importance of inference in establishing mathematical knowledge, and no philosophers subscribe to it.[9]

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[9] Once again I am putting aside Wittgenstein's (1953) notion that all rule following is optional, discussed in Note 1.

Fuller's (1988: 56) second sense of convention is "a practice that has emerged largely without design yet continues to be maintained". Convention in this sense is close to what is implied by participation in Wittgensteinian language games and forms of life, and in Foucault's 'discursive practices'. For to participate in a form of life and its language games is to follow the roles and norms and engage in the expected practices, i.e., to observe the conventions of the form of life. This sense of convention is at least partly implicit, since usually no explicit statement of the conventions will be made, rather participants must infer the conventions from observed behaviours and from others' corrections of their own infractions. This might be termed *historical convention*, since it is based on pre-existing practices. The previous discussion of the role of grammar in language use fits with this conception. In simple terms historical conventions follow practice rather than preceding it, as they do in rational conventions.

Let me now clarify what conventionalism means in terms of epistemology. Social constructivism claims that knowledge and truth are socially constructed. What does this mean? Does it mean that any collection of signs, representations or information making claims about states of affairs 'constructed socially' is equally true? Certainly not, for this would be patently absurd and would lead to the production of contradictions, as when the State of Indiana legislated 22/7 for the value of Pi. Does this claim mean that any historic group reaching agreement about a state of affairs has the right to call its claims true or established knowledge? Here I must agree with Goldman (1995: 1) that "truth must not be equated with consensual belief". For if this criterion was admitted, we would have to acknowledge false and pernicious claims as the truth. For example, the Ku Klux Klan agrees that black persons are inferior to white persons in a number of specifiable ways. Similarly the Nazi's propounded the doctrine of the inferiority of the Jews. The key point here is that such groups had a widespread internal agreement, but that this agreement did not make the claims true. Their dogmas are unfounded claims that social constructivists along with other liberal or humane thinkers reject as pernicious lies. But this raises the question of how else truth is socially constructed.

Truth is not equated with consensual belief. It is not simply created by fiat, agreement, or convention. Instead, truth claims are subjected to criteria for acceptance. Truth claims need to be warranted by something other than simply group agreement for acceptance. In mathematics, a new knowledge claim, a would-be theorem or new result, must be put forward with a proposed warrant for the claim, that is, with a proof. The proof is an argument that is persuasive in establishing the truth, or rather the proven nature of the claim, to experts in the field.[10] The criteria for the acceptance of new theorems do not exist independently of humankind, however, for it requires a group of experts to deploy them. In addition, the criteria are not fully explicit, and I would argue, cannot be rendered fully explicit (Ernest 1999). For they depend in part on the experience and case knowledge of accepted results in mathematics of these experts. Experts serving as journal and conference referees, for it is in such roles that mathematicians act as gatekeepers for the admission of new mathematical knowledge, are not always unanimous, nor even always correct in their judgements, in the eyes of history. But it is through the application of criteria and standards of logic, mathematical practice and rhetorical form that new results are warranted in mathematics (Ernest 1998). It is these second order criteria and standards that represent a central part of the conventional basis of mathematical knowledge. Communities of mathematicians agree on these criteria and standards, not arbitrarily, by rational conventions, but by historical conventions embedded in historical practice. Thus an accepted mathematical result has a persuasive warrant, a proof, that satisfies the appropriate group of experienced mathematicians that it meets current proof standards.

However, the second order criteria and standards for the acceptability of mathematical knowledge are themselves historical conventions, developed organically and historically in communities of mathematicians and transmitted from one generation to the next partially through written criteria and partially through shared meanings and mathematical practices. The vertical variations in such criteria (i.e., over time) are clear, and I claim, cannot be explained by the pushing back of the frontiers of ignorance. What is less commonly acknowledged is the extent of the horizontal variations in such criteria (i.e., over different mathematical specialisms). In one of the few relevant studies Knuth

[10] There are technical issues surrounding the relationship between truth and proof in mathematics foregrounded by Gödel's (1931) seminal incompleteness theorems that for simplicity I shall overlook here but that do not invalidate my argument.

(1985) found wide divergences in the style and rhetorical form of papers in different mathematical topics implying widely varying criteria of acceptance in play. The criteria for acceptance of mathematical results thus vary greatly over time and specialism, and represent accepted historical conventions in place in the different communities of mathematicians involved.

In my view another important factor is also at work. For I claim that there is also a mathematical metaphysics or ideology that sits in place alongside these varying conventions and practices. This asserts the objectivity, universality and certainty of mathematics and its standards and basis. In consequence, mathematicians disregard the variations in conventions, standards and practices of acceptability in mathematics. Instead they see these differences as surface effects that mask an underlying constancy and permanent core of meaning, objectivity and truth.

Foucault's (1980, 1984) concept of a 'regime of truth' concerns the historical truth-status of a socially accepted model, perspective or world-view. When for historical reasons important sectors of a community or society have come to accept such a perspective, and act as if it is true, then a 'regime of truth' prevails. A regime of truth does not concern individually warranted propositions, but an overall metaphysical world-view. Such a perspective may seem as well grounded as a foundationalist account of knowledge, but the basis of such truths is the social acceptance and lived nature of the underlying presuppositions. A regime of truth is hegemonic, and it is held in place by a discursive practice, a set of language games embedded in a form of life, parallel to Wittgenstein's notions.[11]

The metaphysics of mathematics that sees the results of mathematics and the criteria for their acceptance as certain, unchanging and timeless is such a regime of truth. It is a widely shared world-view that overlooks empirical evidence that contradicts it, or explains it away as insignificant. An apparent weakness in this account is that it sounds as if persons have been fooled or even coerced into accepting error or a false account. But a regime of truth enables people to see what *is* as what *has to be*; to identify historically grounded but contingent truths, not

[11] It is also typically maintained by powerful social structures and institutions.

errors, as necessary truths. According to conventionalism, all knowledge presuppositions are the results of different forms of accepted practices, agreement or decision and can all be questioned and reconsidered. Even the fundamental Law of (Non)Contradiction is put aside in a limited way in some modern logics (Routley *et al.* 1982). Mostly the unquestioned basic assumptions in mathematics are an important given, the basis for mathematical research, and certainly this for mathematical applications and education. But sometimes these unquestioned assumptions serve not as buttresses for the certainty and usefulness of mathematics but as epistemological obstacles to further progress. The example of Hamilton's difficulties in questioning the universal applicability of commutativity in algebra, and how this was an obstacle to further progress, is discussed below.

A crucial problem for a conventionalist philosophy of mathematics is to account for mathematical and logical necessity and the certainty of mathematical truths and theorems. Wittgenstein offers an approach to this problem in his later philosophy. Wittgenstein's view of logical and mathematical necessity comes out of his theory of language games. His view is that necessity, such as that of drawing an inference following the laws of deductive logic, which underpins so much of mathematical knowledge, arises from the human agreement in following a rule that is stipulated, presupposed and embedded in a language game.

> The word 'agreement' and the word 'rule' are related to one another, they are cousins. If I teach anyone the use of the one word, he learns the use of the other word with it. (Wittgenstein 1953: 86)

Thus there is no extra-human or objective force that compels anyone to follow a logical rule or to accept the conclusion of a logical deduction. It is rather that participating in certain language games entails accepting certain rules. If one rejects the rule one is repudiating the game as it is understood and played by others. "To obey a rule, to make a report, to give an order, to play a game of chess, are *customs* (uses, institutions)." (Wittgenstein 1953: 81)

Of course the traditional view that logical necessity underpins deduction and rational thought is very firmly entrenched. Wittgenstein anticipates the obvious philosophical objection that rule following in logic and elsewhere stems not from human agreement and but from some essential form of logical necessity, whatever that might be. But even to communicate disagreements about truth, falsity or necessity presupposes that we agree to use the terms comparably in social discourse and life.

> "So you are saying that human agreement decides what is true and what is false?" - It is what human beings *say* that is true and false; they agree in the *language* they use. That is not agreement in opinions but in forms of life. (Wittgenstein 1953: 88).

Agreement in Wittgenstein's sense arises from our participating in shared language games (woven into our forms of life), and does not consist of arbitrarily adopting conventions. This gives us the shared constraints on the meanings of our language, and ultimately leads us to decide what counts as truth and falsehood. Thus the relation among agreement, convention and truth is far more subtle and complex in Wittgenstein's philosophy than in rational conventionalism.

An example, not from Wittgenstein, is as follows. The traditional view of philosophy and logic is that the following logical inference is necessary without qualification: given **A** and **A** **B**, then **B**. My understanding of Wittgenstein's position on this is as follows. Agreeing that this inference is necessary depends on many prior implicit agreements. First of all, the parties to the agreement must all share an understanding of a sophisticated language, written English in this case. This in turn presupposes that the parties are part of a linguistic community and routinely communicate, interact with others and take part in shared social activities. Secondly, the parties (to the agreement) must agree that '**A**' and '**B**' are metalinguistic symbols denoting fixed but arbitrary English propositions, and that every instance of one of them has the same denotation (within an assumed but delimited meaning context). Third, the parties accept the rule of inference *Modus Ponens* as valid (i.e., whenever its premises have the truth as their value, they agree that invariably the conclusion does too.)

Although not an exhaustive analysis, these assumptions show that the logical necessity of the inference depends on shared forms of life (assumption 1) and participation in language games (assumptions 1, 2 and 3). Once these assumptions are made (and most of them you the reader and I the writer as participating members of modern literate and academic society have usually no option but to make[12]) then the conclusion is necessary. Likewise, given another, simpler set of assumptions about the game of Chess, and a particular board configuration, check-mate in two moves is similarly necessary.

Mathematics is the subject par excellence, in which necessity abounds. Once certain assumptions, definitions and rules are accepted the greater part of mathematics does follow by logical inference, i.e., by necessity. But that necessity rests on a set of assumptions that I claim are not themselves necessary in totality. Some crucial elements of mathematical knowledge are contingent truths, handed down from past practice and convention, and consequently the body of mathematical knowledge as a whole is contingent truth.

The social constructivist position is that there is a great deal of stability in a discipline like mathematics while at the same time there are virtually no essential or necessary features handed on down through the millennia. Mathematicians often contrast necessity with arbitrariness, and implicitly argue that if mathematics has no absolute necessity and essential characteristics to it, then it must be arbitrary, and consequently anarchy prevails and anything goes. However as Rorty (1991) has made clear in philosophy, contingency, not arbitrariness, is the opposite of necessity. Since to be arbitrary is to be determined by or arising from whim or caprice rather than judgement or reason, the opposite of this notion is that of being selected or chosen. I wish to argue that mathematical knowledge is based on both contingency, due to socio-historical accident, and deliberate choice by mathematicians, which is elaborated through extensive reasoning and practices into mathematical tradition. Both contingency and selection are active throughout the long history of mathematics. I also wish to argue that the adoption of certain rules of reasoning and consistency in mathematics mean that much of mathematics follows without further choice or accident, by logical necessity, provided we maintain the rules and conventions.

I freely admit that much of mathematics follows by logical necessity from its assumptions and adopted rules of reasoning. However

[12] But note that for philosophical discussion purposes we might temporarily suspend belief in these or any assumptions, i.e., choose to play a different language game.

this does not contradict the conventionalist and anti-absolutist position of social constructivism, for I deny that the rules, reasoning and logical necessity in mathematics are absolute or context-independent. Mathematics consists of language games with very entrenched rules and patterns that are very stable and enduring, but which always remain open to the possibility of change, and in the long term, do change as a totality, if not in every part.

A well known example is commutativity in the multiplication of numbers, so that jk = kj. The $19^{th}$ century mathematician George Peacock framed a law, *The Principle of the Equivalence of Permanent Forms*, that stated that developments in algebra must always respect the underlying arithmetical laws. Thus non-commutativity was virtually unthinkable. Not because of Peacock's dictum, but because of the underlying stable practice that he reflected in his explicit statement. But it was a convention, not a logical necessity. After 20 years of struggle to extend imaginary numbers William Hamilton made breakthrough by rejecting jk = kj, and putting jk = -kj instead. This led to the important Theory of Quaternions. In so doing, Hamilton respected and conserved many of the laws of algebra, but also made extensions and significant changes (Pickering 1995). He overcame what Bachelard (1951) called an epistemological obstacle, a received, conventional part of the conceptual apparatus of mathematics that obstructed further progress; a contingency that appeared a necessity.

Such dramatic changes in which past strictures are overturned to develop a new and fruitful theories may not happen every day in the history of mathematics. But they do happen regularly in children's development in school mathematics. It is commonplace for teachers to extend mathematical topics requiring the negation of existing rules and the change of underlying meanings, through the adoption of new rules. For example, for a young child mastering elementary calculation, the task 3-4 is impossible. But later it has a determinate answer: 3-4 = -1. Similarly 3 divided by 4 (3/4) is at first an impossible task. Later it is not only a possible task, but ¾ names the answer to it, i.e., becomes a new kind of semiotic object, a fractional numeral. In early multiplication tasks children learn implicitly or explicitly that "multiplying always makes bigger". Later when the domain of numbers they operate on is expanded to include fractional and decimal numbers (i.e., Rationals), or even just zero, this rule is contradicted. The difficulties caused to learners by these changes in definitions and conventions are well known.

In these and many comparable cases the rule changes are necessitated by changes in the underlying meaning of the operations. Thus subtraction, initially, is usually understood in enactive or metaphoric terms as resulting from the partitioning of a collection of concrete objects and the removal of one part. Hence 3-4 *is* impossible, it is not just a matter of learner ignorance. Subsequently in learner development subtraction is commonly understood more structurally as the inverse of addition applied to an enlarged and more abstract domain of numbers. Hence since 3-3 = 0, 3-4 = -1. But this is a new mathematical system, which represents an intellectual advance that took humankind hundreds of years to make and to accept. It is very likely that the later more abstract meaning of subtraction cannot be developed without the earlier concrete meaning, so the apparent contradiction is unavoidable. In examples such as these, the student has to 'unlearn', that is relinquish something already learned, in order to make further progress. Following Bachelard, these problems have also been termed epistemological obstacles (Brousseau 1997, Sierpinska 1987).

It is through immersion and participation in the practices of first learning mathematics, and later doing research mathematics, that mathematicians are enculturated into mathematical forms of life with their tacit rules, conventions and knowledge. These contingent features, extended and elaborated through rigorous reasoning and proof, give mathematicians a sense of the necessity of their subject. They also enculturate them into the standards and criteria for the acceptance of new mathematical knowledge. But if only one convention or contingency is woven into this knowledge, and I claim there are many more than one, then mathematics as a whole is conventional.

## 4. Conclusion

The version of social constructivism that I have been discussing is a naturalistic philosophy of mathematics that aims to provide an account of mathematics as it is practised, cognizant of both the social structures within the mathematical community and the historical development of the discipline. I have argued that in a particular sense of the term it is nominalist, because it regards mathematical objects as signs deployed within semiotic systems with sign rules and meanings. I have not denied that abstract objects exist, just located them in the realm of culture, alongside money, literature, and other human institutions and artifacts.

I have argued that mathematical knowledge is conventional in the sense that it is warranted by the rules of mathematics and the mathematicians' understandings of logical necessity. However, I have claimed that these rules, and mathematicians decisions of acceptability based on them, is itself partly a result of historical contingency. By subscribing to these limitations and deviations from the traditional ideology of the purity, objectivity, and perfection of mathematics I aim to reclaim mathematics from the idealists. Bringing mathematics back down to earth, to the mundane reality of lived human life, is not to denigrate or besmirch it. Ironically, the aim is to offer a more accurate and a truer picture of mathematics as a part of lived human experience.

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