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THE INCIDENCE OF ANALOGICAL PROCEDURES IN THE EMERGENCE OF MATHEMATICAL CONCEPTS. NEWTON AND LEIBNIZ: A CASE STUDY.

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1. Introduction

Analogies are not sufficient to justify the knowledge they support. Since analogical reasoning is not conclusive, it always possible to find something else to change the orientation of thought.

Nevertheless, they constitute powerful ampliative heuristic principles to start and guide a non-blinded search to find more satisfactory justifications about what we presume true.

But also, they could secure our partial convictions and serve as stepping-stones to reach higher degrees of confiability of the started hypothesis.

A very first good idea, instead of being just considered as a non-conclusive obstacle, may open roads for further investigation if we can count on them, on the way of looking for reasons that support the previous predictions.

Analogies are useful not only because they allow us to increase the possible different perspective alternatives, but also because they give clues to bet in a specific course of action.

In what follows, there's an attempt to show how Leibniz and Newton apply analogies - in very distinct ways- in the search of knowledge. It is the purpose of this paper to show the specific differences between them, although both reached a method that could solve eventually the same mathematical problems. The analysis of the incidence of analogies in the development of Calculus will be restricted in this paper to the differential side of the coin, leaving aside the Integral Method and the relation between both of them, either in Leibniz´s as in Newton´s case.

2. Leibniz´s mathematical approach

2.1 Mathematics in context

There are several decisive elements that influence in the way in which Leibniz conceptualizes the trilogy Nature-God-Mathematics. Three poles of his theoretical frame, mutually dependable, absolutely inseparable. It would be too artificial and difficult to give an explanation of his characterization of Mathematics without introducing the other two aspects. The mentioned elements can be pointed out synthetically:

2.1.1 As a typical representative of Modern Age, Leibniz distinguishes clearly and crisply two ambits: the *empirical*, associated to sensitive perceptions from which only the factual *contingent truths* arise, and the *rational*, linked to the intelligible aspects from which the necessary and eternal truths *truths of reason* emerge, within which the Mathematical theorems are found.

2.1.2 Leibniz structures his thoughts around *principles* he formulates and that human understanding in general is naturally and necessarily forced to obey. He considers these principles articulate among each other in a very solid and coherent way. One of them, the Principle of Continuity is especially pertinent to the following discussion. Leibniz presents at least three versions:

Principle of Continuity, version 1:

Nothing takes place all at once, Nature never makes a leap.

Principle of Continuity, version 2:

"...When the difference can be diminished below any magnitude contained in the data or in what is proposed, it must also be possible to diminish it below any given magnitude what is looked for or what results." (Russell, 1900)

Principle of Continuity, version 3:

"When the cases (or what is given) continuously approximate one to the other, and end up by interpenetrating each other, the consequences or events (or what is seeked) must also do so." (op.cit., 1900)

Leibniz considered that the totality of principles operated generating a system oriented to unify criteria of order, perfection, coherence, and continuity, resulted as a consequence of the interaction of them, in a conception of a rigorous, precise, clear and complete Mathematics, as a *perfect divine product.*

2.1.3 The third decisive element is *language*, which plays a fundamental role as a means to reach Nature and Math through the human mind, finite limited minds that share with God only a small portion that is needed to comprehend what's going on everywhere.

Leibniz finds in Math a very good model of thought that we, humans, can always arrive at. He believes that Math presents unsuspected features that brings humans nearer to the infinite wisdom of God, and that *can be used in other contexts analogically.*

This responds to a global project of unification through the search of a common language for all the scientific disciplines and also for all the human activities, a UNIVERSAL CHARACTERISTIC that would signify an advance in every area.

Leibniz thought of the possibility of constructing a calculus of reasoning based on the best of all the possible languages, called by him a UNIVERSAL LANGUAGE (LINGUA UNIVERSALIS), that would interpret the special relations he thought that existed between the different characters. These relations would emerge no matter what type of signs were used for the sake of expressing the ideas, i.e., some proportions and equivalencies exist between the underlying relations that could give the right meaning of things.

This universal language, formed by characters, signs or words, would not be entirely arbitrary, though Leibniz didn't believe that a correspondence existed between words and things signified by them. Nevertheless, truth is neither arbitrary nor conventional, because some kind of proportional relation or order exists in the characters, *similar* to what occurs between things.

Analogies will occupy here a significant place in Leibniz´s method of invention and development of Math, as we shall try to show through this paper.

2.2 The relationship part-whole

The dichotomy continuum–discrete in Leibniz is based on another demarcation previous to this one: there is, on one hand the *actual*, i.e. the real world of concrete nature and experience, and on the other, the *possible*. This last category are includes the world of Mathematics and the eternal truths.

This demarcation, in fact, characterizes two philosophical points of view present in Modernity: Empiricism and Rationalism respectively, the latter within which Leibniz has definitely enrolled himself.

Concrete objects, according to Leibniz, are always discrete wholes, i.e., the parts in them are distinguishable and independent among each other; whereas in the case of Mathematics, objects are continuous and their parts are constituted from the relations that can be established with the whole. They are significant only in relation to the whole. This turns them into elements which are indiscernible by themselves, and which lack a unique definition outside the context of the whole.

The continuum is uniform, the elements in it are indistinguishable, homogeneous, and therefore comparable.

> I generally say... that elements are *homogeneous* when they can be made similar through a transformation of one into the other, as, e.g., a curve and a straight line. (Wiener, 1951)

One *part* then, is a homogeneous ingredient contained in the whole. But in Nature, instead, it is not possible to find two different elements sharing all their attributes.

> I have observed... that, by virtue of imperceptible variations, two individual things can not be perfectly similar, and must always differ in something else than in number. (Russell, 1900)

This homogeneity property and the difference stated here are crucial to clarify the distinction between the terms "discrete" and "continuum", specifically in reference to their intervention in the characterization of *infinitesimal magnitudes*.

2.3 Perceptions vs. Apperceptions

In an attempt to go further in the clarification of *continuum,* Leibniz applies a series of interesting *analogies*[1] in order to understand its nature. For that purpose, he makes use of a trick that, in some way, combines the concrete with the abstract.

Let us remember that *apparently*, in the frame of his theoretical apparatus this was impossible. However, this is not so. It is strange how this concrete case, axis of the controversy continuum-discrete, leads him to imagine the continuum as a process of successive perceptions. However, he does not understand the term *perception* from the point of view of Psychology, a discipline in which this notion consists basically of the collection and elaboration of data provided by the senses. To characterize the latter, Leibniz uses the expression *apperception*, which means "the *expression* of the multitude in one unit." (Russell, 1900)

The human mind acts integrating the diversity of sensations, reflecting on them and constituting a unity within that plurality. Every object of knowledge, despite its simple appearance, holds a variety of aspects that are often unnoticed by the subject's attention. They are what he is going to call *"minute petite perceptions"*, that appear undistinguishable, confused, with no precise boundaries, undetermined.

If the aim pursued were *"to capture perceptively"* (the image of) continuum, it would not be noticed perhaps that this would lead through a process of "minute perceptions". According to Leibniz, one would probably think that the continuum might only be fully recognized in a single instantaneous act, despite the fact that this is only a result of that process perceived as absent.

But it is not that something *suddenly* occurs, or that in an instant, an interruption of the continuum is produced, but that there is a gradual and successive occurrence of things that, upon the instance of *saturation* of some kind, or of *accumulation* of events, the change is produced.

But this is only apparent: there is no such abrupt change, but this incessant aggregation of very small elements, almost imperceptible in which a *discernment threshold* has been reached, where we suddenly

[1] Here, the concept of *analogy* is used playing the role of *metaphors* that provide a feedback to the information previously given and forthcoming, i.e., they play an anticipatory and ampliative function of the compared concept in question.

perceive this whole as arising ex nihilo.

The point here is that these *"minute petites perceptions"* are undetermined or confused, and would not make any sense if it were possible to isolate them: this is precisely the problem, there is no such possibility of discrimination. This is *the key and the enigma of the continuum.*

The notion of infinite is strongly associated here to the type of process of addition or accumulation or junction: "These impressions... are not strong enough..." (Wiener, 1951). If we could consider them separately, in isolation, "... a hundred thousand *nothings* cannot make something...", "... because the impressions are either *too slight* or in *too great a number* or *too even*, so that have nothing sufficient to distinguish them one from the other...". But instead "... joined to others, they do not fail to produce their effect and to make themselves felt at least confusedly in the mass." (op.cit., 1951)

2.4 The creative role of language

These considerations lead us then to Leibniz' first publication where the notion of *differential* is found (in 1684 in *Acta Eruditorum*), together with the *method of calculus* that he produced based on this concept.

There he presents the *algorithm* of this calculus, which is denominated *Differential Calculus*. Any type of curves (including the ones called "transcendental" that had never been dealt with other than geometrically before, but without an analytic precision as arises now) is associated with certain *differential equations*, in order to find tangents to points of the curve. To apply this method,

> ... we have only to keep in mind that to find a *tangent* means to draw a line that connects two points of the curve at an *infinitely small distance*, or the continued side of a *polygon with an infinite number of angles,* which for us *takes the place* of the *curve.* This infinitely small distance can always be *expressed* by a known differential like *dv*, or by a relation to it, that is, by some known tangent. (Struik, 1986)

In the specific case of this mentioned paper, the generic curve line there dealt with, can be visualized in a way similar to the one applied previously with the continuum.

Thus, we do not consider the curve line as formed by *material* points. It is certainly possible to distinguish *mathematical points* on the curve, which are considered by Leibniz as *exact,* like *points of view for expressing the Universe*, only *modalities* (Wiener, 1951).

Points cannot add up or arrange to produce the totality of the line as a result, and, in this sense, there are no more than "limit-entities", abstractions.

However, what counts here is to consider the "undetermined *parts*" of the (continuous) curve, and, due to the homogeneity property, the mathematical points do not fulfill the requirement of being *part*, given the fact that they are of a lesser magnitude order.

The point here is that generally, in this theoretical frame, the parts must be similar to the whole they constitute. Thus, a part of a straight line must also be a straight line. We can notice here that the way of referring to a (bounded) segment as a straight line itself, which was usual in ancient Greek times, is still in force: "... the straight line is the shortest line from one point to the other ..."(op. cit., 1951).

In this way, it follows that the curve line involves innumerable minute particles or straight line segments, which contain and *express* in themselves all the properties of the curve line.

The term *"expression"* is also a technical term in Leibniz' terminology:

> One thing *expresses* another when there is a *constant and regular relationship* between what can be said about one and about the other. In the same way as a projection in perspective *expresses* its original. (Russell, 1900)
> ...[The] means of expression are varied; for example,... letters express numbers, an algebraic equation expresses a circle or some other figure... (Wiener, 1951)

There are at least two reasons due to which the notion of expression gains value in Leibniz´s work: the first of them is in reference to the importance of *the notation in itself* for mathematical production ; the second is related to the role played by analogical procedures in the conceptualization of the notion of *expression* given by Leibniz.

Regarding the first item, the usage of certain symbols and not of others can contribute not only to fasten the understanding of the theory in use, but it can also turn out better for the heuristic purpose of generating new knowledge: a conveniently well chosen notation can concentrate and synthesize thought processes that simplify and facilitate the display of ideas.

It is important to stress here what has already been mentioned in a previous section of this work, regarding the power Leibniz thinks a good sign system has, a "characteristic", to serve as a tool in the *Art of Invention*, as well as in the recognition of concepts and problems in question. In a letter to L'Hôpital, dated April 28, 1683, he says:

> One of the secrets of analysis consists in the characteristic, that is, in the art of skillful employment of the available signs... (Cajori, 1993).

2.5 Analogies in mathematics

In terms of this heuristic role attributed to notation by Leibniz, and dealing now with the second item related to the notion of expression, the way in which expressions are related to analogies is important.

It has already been mentioned that the expressions of things consist of "copies" or linguistic imitations of other things, or, in other words, in functions that co-relate one to the other.

Now, there are situations in which this correspondence process is not, or cannot be carried out in a totally perfect way. Even in these cases, Leibniz considers it sensible to proceed to the application of an *approach function*, "as long as certain *analogy* of conditions is kept." (Russell, 1900)

Specifically, this type of *expressions* can be carried out in Mathematics, since it is there that very particular circumstances are generated, under which this idea of approximation is materialized and, consequently, it is possible to apply the Principle of Continuity previously mentioned, in which the analogy turns out to be natural. The relationship between ANALOGY and CONTINUITY PRINCIPLE, will be dealt with further on.

Indeed, in the empirical, real, natural world, there is only *determination* ; everything is always absolutely specified: "*...indetermination is the essence of continuity*." (Leibniz(b), 1705)

It is convenient to analyze now the way in which it is showed here that such interconnection is produced from analogies.

By stating that the previously mentioned polygon *takes the place* of the curve, Leibniz means that it *expresses* the polygonal with the curve, i.e., coming back to the afore said, it substitutes an element by the other through a relation of correspondence.

Now, this exchange process implies being able to specify the conditions of such change:

> ... it is because these means of expression *have something in common* with the conditions expressed and studied, that we can come to know the corresponding properties of the thing expressed. Hence, evidently the means of expression need not be similar to the thing expressed, so long as a certain *ANALOGY* holds among the conditions in both. (Wiener, 1951)

So, the *clue* of how such simplifications of the curve in the polygonal are produced is *by ANALOGICAL TRANSFERENCES.*

And this analogy is plausible because it is possible to find imperfect similes of mathematical objects in nature that can be compared with a certain degree of approximation, even when we understand or know the difference with total accuracy.

> ... Although the idea of a circle is not exactly like the circle, we may yet infer from the idea truths which experience would undoubtedly confirm concerning the true circle. (op. cit, 1951)

If it can be done with the circle, then why is it not possible to do it analogically with the curves ?, i.e., conceive them as composed by many very small segments, chained among them, constituting in fact a polygonal. This is nothing else than *expressing* it in the sense of perceiving the plurality present in the unity of the continuum of the curve, where correlations of the *indeterminate* parts with the curve as a whole exist.

Now, what guarantees that this is a good analogy is the Law of Continuity in version 2. From now on it will be called here also *Leibniz´s Principle of Analogical Transference.*

In which way does this law give the necessary guarantee? Under which conditions?

Answer: *Similar cases behave in similar ways, but must be similar enough in the relevant aspects to be disregarded in those in which they differ.*

In other words, and applied to the case (of the curve) dealt with here, the criteria of distinction between the curve and its analog, the polygon, that allows to accept that the simplification is proper, is this second variation of the principle of continuity 1. It very much recalls the present definition of *Cauchy sequence*, which was instituted after Leibniz, in order to give solid basis to the Calculus through the formal mathematical notion of *limit*.

However, it is held here, Leibniz will disregard the idea of *process to the limit*, in favor of a methodology based on the abstractions arising from its analogies. The concept of "relation" allows him to deal with abstractions via analogies.

For Leibniz, *an analogical transformation is a relation, i.e. an isolated result extrapolated from two or more elements, which are comparable by similarity*. This result does not belong to the concrete order any more, but to another category or level, and that characterizes a mental process of agreement of elements not totally identical.

Analogical transference can be made effective given the fact that the two different domains that fall into the comparison only do so in a group of common properties that have been able to separate the remaining characteristics that both concepts at stake might independently have, in an abstractive process.

In this way, in the example followed as a model, the mathematical curve matches a finite polygon (with a finite number of segments). The straight line, tangent to the polygon, in a point interior to some of its segments, is simply the extended straight line that comprises such segment. The abstractive reduction made to the segment to the point in which its tangent has to be found, allows to analogically transfer the mentioned tangent straight line to the point of the curve.

The importance of characterizing analogies as certain type of relations lies in the advantages of the latter in order to *create* new concepts.

The entities created by a relation of similarity are *useful*, and therefore they are then *well founded fictions*, because they would let us understand the world, and if we were so smart, so cultivated, Leibniz thinks we could even dominate the *ART OF INVENTION, something* only God masters to the perfection.

2.6 Inductions out of mathematics

It's important to ask now at last why Leibniz chooses as a model of cognitive search the one associated to analogies and abstractions and not, for example, some kind of incomplete induction or generalization. Answer: Leibniz associates the inductive processes with experience and, therefore, keeping steady in his rationalist convictions, he rejects them insofar as they are related to mathematical truths.

For Leibniz, induction is greatly linked to "perceptive" processes (not just "apperceptive", let us remember this was a technical term in his theory), and that makes him also very strict as regards the relationship between induction and analogy. It would be very difficult at present to separate completely these two concepts, which seem to have many links that make them, in some sense, complementary processes. This is what we will see next in Newton's approach.

In Leibniz however, analogy corresponds to abstract processes in search of necessary, eternal truths, and therefore is purely mental; while inductions are purely empiric generalizations guided by the prosecution of contingent truths and, as is well known that rationalists like Leibniz totally delimit these ambits, this would lead him to discard any eventual connection.

What makes someone think that Leibniz was not referring to "limits of processes" in a dynamic sense, like for example it was Newton's approach, instead of a static abstraction, i.e., a reduction by successive elimination of properties, an idealization obtained by abstracting from divisions in smaller parts, a synchronous and not diachronic point of view ?

It's his permanent philosophical effort to update the dycothomy between the empiric and the ideal (even though he accepts a relationship between both sources). Leibniz would not think of this process being obtained by aggregating qualities abstracted from many unitary material components, meaning an INDUCTIVE LEAP, but by abstracting features that they all share, i.e. an ANALOGICAL LEAP.

This means that Leibniz would throw away all limit processes that imply the proper dynamics concerning transit by different stages. In this way, starting from a material segment of a polygon that can be endlessly subdivided producing a constant decrease of successive lengths, this would lead us to suppose that the process would also converge to a *material* point.

Leibniz would reject this last conclusion: definitely there is not an ultimate material element or *atom* in these infinitely empirical processes, failing this kind of generalizations.

What counts here in the case of induction is that it's not possible to bridge between several concrete entities on one side, to an abstract one on the other side. We must stay at a concrete level (that's what is allowed in the inductive fields), and so this is the CLUE from which we can abstract an ideal mathematical point starting from a material curve.

3. Newton's mathematical inquiry

3.1 General perspective

Newton's approach concerning infinitesimals corresponds to a more general project that comprises the entire domain of Mathematics and Physics, as well as Philosophy. His "Natural Philosophy" has peculiar connotations provided by the incorporation of mathematical techniques that has nothing to do with the typical position that colleagues of his time support, related to the search of hypothetical causes in Nature.

Instead of that, his investigations focalizes mathematically on phenomena and how they contribute in capturing the general principles that govern them.

> These principles I consider, not as *occult qualities* supposed to result from the specific forms of things, but as general laws of nature by which the things themselves are formed, their truth appearing to us by phenomena, though their causes be not yet discovered. For these are *manifest qualities*, and their causes only are occult... such occult qualities put a stop to the improvement of natural philosophy, and therefore of late years have been rejected. To tell us that every species of things is endowed with an occult specific quality by which it acts and produces manifest effects is to tell us nothing, but to derive two or three general principles of motion from phenomena, and afterward to tell us the properties and actions of all corporal things follow from those manifest principles, would be a very great step in Philosophy, though the causes of those principles were not yet discovered. (Thayer, 1953)

Newton advocates the idea that Nature is governed by these principles of unity and consistency. He places Mathematics as an integral part of Philosophy, no less than Physics. Everything should be coherently connected having a bearing on the epistemological accounts: the results of inquiry in one context or discipline may be relevant to the search and appraisal of results in any other.

He pursues a reductionist purpose of seeking for a common unified explanation as a starting point to try to express everything by means of a unique tool that relates the unknown with the acquired scientific experience.

In the case of the new concepts that emerge via his Method of Fluxions, for the sake of acceptability it is necessary to understand how the supervenient phenomena and their mathematical concepts could be appropriately related to the theory still in force.

This epistemic attitude, together with the fact that the old problems obtain clarity in the light of the developments given by the new theory, recognizes his trend as reliable, though it doesn't achieve a solid status as the mathematical standards require and would be reached much later when the notion of "limit" would be established.

Newton's success in the treatment of the subject at stake depends on his original dynamical operational proposal based on the model extracted and constructed analogically from Mechanics. The fluid relation he conceives between Physics and Mathematics allows him to extrapolate techniques from one discipline and apply them to the other. This mutual interdisciplinary collaboration enriches his approach, introducing analogies as a new strategic methodological orientation.

3.2 Cognitive levels in mathematical knowledge

Newton's "Natural Philosophy" is characterized by a combination of physical matters with mathematical demonstrations, leaving aside the problems concerning the causes of phenomena. The distinction between Mathematics and Physics is not preserved under Newton's interpretation, insofar as Mathematics contributes in his works to model the knowledge of Nature, and not only providing a set of tools that could be applied - without intervention of mathematical ideas- to solve physical problems, as we shall see upon the construction of the cognitive levels involved in mathematical knowledge.

This later alternative relies on a position alien to Newton, that views Mathematics as an instrumental science that has nothing to do with "natural" matters, while the former allows Newton to make use of mathematical concepts and arguments in physical contexts: what is treated by mathematical reasoning, must be considered belonging to a mathematical field.

The close link Newton presumes that Mathematics and Physics possess - especially Geometry with Mechanics - acquires relevance when he endeavors to clarify the dynamical status that attain geometrical magnitudes, and the way this characterization contributes to explain the relationship between the discrete and the continuum.

It is possible to classify - based on texts extracted from the Principia - Newton's processes of acquisition of mathematical knowledge in three levels: instrumental, postulational and demonstrational. They will be described in strict relation to the development of his Fluxion's Method, but they can be generalized to other branches of Mathematics and Physics, as Newton shows in other parts of his work.

Instrumental Level (L_1)

First of all, mathematics requires learning certain skills, to be trained in the use of the tools that characterize its practice, to become proficient in the manipulation of geometrical constructions, e.g., to be capable to draw lines and circles:

> ...the description of right lines and circles, upon which geometry is founded, belongs to mechanics. Geometry does not teach us to draw these lines, but requires them to be drawn, for it requires that the learner should first be taught to describe these accurately before he enters upon geometry, then it show how by these operations problems may be solved. To describe right lines and circles are problems, but not geometrical problems. The solution of these problems is required from mechanics, and by geometry the use of them, when so solved, is shown... (Motte/Cajori, 1729/1934)

Newton draws a sharp line between the first two levels. This distinction is supported on another division: the one between Geometry and Mechanics. Newton also uses an old classification that distinguishes Rational Mechanics from Practical or Manual Mechanics. It could be possible to add to this proposed classification an L_0 level dealing with this latter manual area, but it wouldn't affect Mathematics.

Rational Mechanics is the discipline that accurately describes mathematical mechanisms that are going to be used at the next level to formulate postulates.

L_1 is described as a blind level in Newton's scale of cognitive appraisal, where one only needs to obtain expertness in the use of mathematical techniques without trying to ask about the general consequences that might produce the application of them in one or other concrete situation. The acquisition of these abilities is a necessary step in reaching the next level.

Postulational Level (L_2) and Demonstrational Level (L_3)

Newton talks first about deriving principles from phenomena and second, how properties and actions follow from these manifest principles. The first of these two positions will define the next level while the latter deals with the third one.

Once a practitioner knows how to master mathematical operations, he/she is able to generate postulates, therefore dealing with the possibility of existence of mathematical entities, relations and results, by means of "mechanical" constructions, which should be confirmed in a new step.

This signifies that the results, properties or principles are *potential,* not necessarily developed, and will eventually be proved through constructions.

The importance of the recognition of their "potential" epistemic status is that this allows them to be suitable for use in arguments, i.e., potentiality entails plausibility. They could provide the sufficient evidence that is necessary to initiate the search of their truth. So they could go beyond the instrumental level and are prepared for further assessment.

By this stage it is possible to imagine an "empiricist" Newton in Mathematics. Certainly this is not the case. Mathematical "experience" consists, in Newton's account, in the geometrical practice of constructions of lines, circles and other geometrical entities or relations. This doesn't mean necessarily that Newton adopts the alleged position. In effect, the kind of experience at stake corresponds to activity that belongs to the first instructional level of Rational Mechanics, that assimilates to a repetitive practice oriented to the correction, mastery and accuracy of the techniques, and has nothing to do with the postulational or the demonstrational levels, where the general affirmations could be formulated and proved. These two other levels are the ones in charge of the acquisition of "real" mathematical knowledge. The apprenticeship of the mathematical tools doesn't contribute to the "universal" characteristic knowledge should have. Only these two other levels matter in order to generate modes of mathematical thought. Moreover, as clarified below, analogical inferences will intervene in a creative way only from the second stage on, though it's possible to find similarities among operations, but without the heuristic power to extend their applications to general cases.

3.3 Against language

Newton is a strong defender of an inescapable division between language and the world of experiences, unlike Leibniz, as we saw above, who presumes that there is a universal code from which everything could be stated explicitly. In the case of Leibniz, unity is provided by language, while in Newton's account Nature is in charge of that task.

Newton posits language at the same reduced rank that was given to the mathematical techniques from level L_1, inasmuch as this critique to language relies on the same basis of his position about mathematical tools.

In effect, at the instructional level there isn't creativity, and the similarities that could be found appealing to constructions do not involve generalizations from analogous cases. Sometimes language limits the boundaries of search. In order to model creatively we need to surpass it. Underlying Newton's critique to Leibniz, is this specific role that the latter claims about the relevant role that language possesses in the processes of discovery, firmly opposing to him with respect to the argumentative force expressions could acquire.

Moreover, the application of techniques, either from Mathematics or from language, for example the mathematical language of equations, not necessarily could better help the understanding of the phenomena they attempt to describe.

> Equations are expressions of Arithmetical Computation, and properly have no place in Geometry, except as far as quantities truly geometrical (that is, lines, surfaces, solids, and proportions) may be said to be some equal to others. Multiplications, divisions, and such sorts of computations, are newly received into Geometry, and that unwarily, and contrary to the first design of this Science... therefore, these two sciences ought not to be confounded.
> The Ancients did so industriously distinguish them from one another, that they never introduced arithmetical terms into Geometry. And the

> Moderns, by confounding both, have lost the simplicity in which all the elegance of Geometry consists... (Fauvel & Gray, 1987)

No matter which could be the description style we adopted, what really matters is the cognitive process behind the representational system we use to signify. To apprehend a phenomenon consists in finding the principles that govern the production of it, independently of the procedure that is used to express it.

> ...It is not the equation, but the description that makes the curve to be a geometrical one. The circle is a geometrical line, not because it may be expressed by an equation, but because its description is a Postulate. It is not the simplicity of the equation, but the easiness of the description, which is to determine the choice of our lines for the construction of problems. (op. cit., 1987)

Newton distinguishes "contradiction in terminis" from "contradiction in Nature", trying to minimize the importance of the former and attempting to explain how Nature could produce the latter. A special case where he had to elaborate a reply to Mr. Bentley through a letter beard on this subject:

> ... whereas many ancient philosophers and others, as well theists as atheists, have all allowed that there may be worlds and parcels of matters innumerable or infinite, you deny this by representing it as absurd as there should be positively an infinite arithmetical sum or number, which is a contradiction *in terminis,* but you do not prove it as absurd. Neither do you prove that what men mean by an infinite sum or number is a contradiction in nature, for a contradiction *in terminis* implies no more than an impropriety of speech. *Those things which men understand by improper and contradictious phrases may be sometimes really in nature without any contradiction at all*: a silver inkhorn, a paper lantern, an iron whetstone, are absurd phrases, yet the things signified thereby are really in nature... (Thayer, 1953).

> ...if any man shall take the words 'number' and 'sum' in a larger sense, so as to understand thereby things which, in the proper way of speaking, are numberless and sumless (as you seem to do when you allow an infinite number of points in a line), I could readily allow him the use of the contradictious phrases of 'innumerable number' or

'sumless sum´ without inferring from thence any absurdity in the thing he means by those phrases. [2] (op. cit., 1953).

As these quotations show, Newton accepts that it is possible to arrive at these paradoxical concepts in science. Understanding them could lead to the emergence of new concepts. R. S. Westfall quotes an example where Newton puts in practice the idea of contradictions in Nature: the concept of *vis inertiae,* "the force of inertia, or perhaps, in a free translation, the activity of inactivity" (Westfall, 1980). The most interesting problems in the development of Calculus are characterised as inherent contradictions. The next section will be devoted to Newton's solution, according to his method of Fluxions.

3.4 Modes of perception

Since Antiquity, the comprehension of the relationship between the discrete and the continuum was a difficult task. Newton not only captures the paradoxical condition of this problem, but could solve it by using a different conception of natural experience, that would lead him to propose a convenient vinculation between Geometry and Mechanics, and a special way to interpret perceptual change processes.

He thinks Nature serves as an heuristic devise to produce analogical models that could be applied on mathematical inquires. In order to describe mathematically the continuum, Newton appeals to a physical treatment of notions as *change* and *generating processes*, using the concept of time as a clue element which would play a fundamental role on the notions of curves and geometrical figures.

Newton's own words stating the geometrical problem concerning *evanescent divisible quantities* were:

> ...You are not to suppose that quantities of any determinate magnitude are meant, but such as are conceived to be always diminished without end (Motte/Cajori, 1729/1934)
> Perhaps it may be objected, that there is not ultimate proportion of evanescent ; because the proportion, before the quantities have van-

[2] Letter from Newton, Cambridge, February 25, 1692/3.

ished, is not the ultimate, and when they are vanished, is none. But by the same argument it may be alleged that a body arriving at a certain place, and there stopping, has no ultimate velocity ; because the velocity, before the body comes to the place, is not its ultimate velocity ; when it has arrived, there is none. But the answer is easy ; for the ultimate velocity is meant that with which the body is moved, neither before it arrives at its last place and the motion ceases, nor after, but at the very instant it arrives ; that is, that velocity with which the body arrives at its last place, and with which the motion ceases.. And, in a like manner, by the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities not before they banish, nor afterwards, but with which they banish. (op. cit., 1729/1934)

There is a limit which the velocity at the end of the motion may attain, but not exceed. This is the ultimate velocity. And there is the like limit in all quantities and proportions that begin and cease to be. And since such limits are certain and definite, to determine the same is a problem strictly geometrical. (op. cit., 1729/1934)

...Those ultimate ratios with which quantities vanish are not truly the ratios of ultimate quantities, but limuts towards which the ratios of quantities decreasing without limit do always converge ; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished in infinitum. (op. cit., 1729/1934)

The ratios of quantities here mentioned by Newton do appear to move towards/to decrease attaining to the limit stage, and also, at a given time exactly when they presume to reach that stage, might appear to stay still/to vanish. But this process never finishes.

The same phenomenon provokes a contradiction in nature, as the *vis inertiae* quoted previously by Westfall, and mentioned before. These magnitudes look like both moving and not moving at the same time, and this situation seems to be suspicious. But it is not. In effect: Newton's magnitudes don't effectively have the property of moving, but humans could perceive the processes involve in change analogically, as if it really had happened. Most precisely: one can have an experience with two apparent conflicting representations of it, insofar as there are involved two different perceptual processes. There's a duality enclosed in the same phenomenon, due to the possibility of constructing two interpretations of it. Newton plays with this ambiguity for the sake of argument through his Fluxion Calculus. He was dealing not with ratios of determinate parts, but always with the seemingly intangible limits of ratios.

These combined perceptual processes take place when we consider "the velocity at a motion state but at the very instant when the motion ceases", in mechanical terms, or, otherwise geometrically, applying the mentioned analogy, when we look at "the ratio of quantities at a non-banish state, but at the very moment(point) the ratio banish".

The emerging naive idea of *limit* of an infinite processes Newton proposes, is supported by the confidence on inductive approximations as long as there is a reason to believe they would converge to a determinate value, and this reliability is provided by the so-called Lemma I, section I, Book I: The Motion of Bodies, from the Principia:

> Quantities, and the rates of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal. (op.cit., 1729/1934)

This Lemma provides a warranty test to the successful convergence of the mentioned limit processes, reminding Leibniz´s Principle of Continuity, version 2 as well as the subsequent Cauchy Sequence's Criteria. This is not fortuitous and it is strongly related to a plausibility criteria that will be proposed below concerning analogical procedures.

3.5 Analogy and induction: Newton's interpretation

The following quotation corresponds to an extract of a letter from Newton to Mr. L´Abbé Conti, dated February 26, 1716, London, containing a severe critique addressed to Leibniz, where Newton exposes briefly his point of view concerning experience, hypotheses and induction:

> If it would be required to inspect his philosophy, it couldn't be difficult to show that he had distorted from ordinary use the signification of words: since, for example, he called *miracles* to the things that occur in the ordinary course of Nature ; he named *occult qualities* to things whose causes are not unknown...It couldn't be shown that this preestablished harmony is a genuine miracle, and that it is opposite to the experience of all men ;...it could be fairly reproached him for preferring hypotheses to arguments by induction, extracted from experience; that he accused me for having opinions that are entirely not mines, and

> that instead of proposing questions whose examination would be submitted to experience before they were admitted in Philosophy, he proposed hypotheses he wanted that were admitted before being examined. (Leibniz(a), 1716)

His position can be summarized by these characteristics:
(1) In order to be admitted in Philosophy, a matter should be submitted to inspection provided by experience.
(2) Experience should be universally shared.
(3) Hypotheses are not accepted in Philosophy: they deal with causes and Philosophy isn't involved with causes.
(4) Philosophy admits inductive arguments, rooted in experience.

The point here is what Newton means by "inductive arguments". The kind of induction he is dealing with allowed that the starting affirmations from these arguments don't need to be general, even when it is suitable that one can associate with them a *certain* generality. In effect, the analysis of a unique particular case in study could show certain regularities or properties instanciated on it that can be generalized to the total reference population. If there are sufficient reasons to think that the requested information involved in the examination of that single case, constitutes a typical or representative sample of the totality, then we can assume that the argument about this class is acceptable, without requiring more cases that confirm the suspected properties.

This phenomenical inspection allows to formulate *postulates*, as were mentioned in a previous section of this paper. The analogies involved in the establishment of these postulates are not mere *illustrations* of similar phenomena but they acquire relevance on the discovery and modeling of them, and also contribute in the processes of justification, although they don't represent a real formalized way of producing mathematical knowledge as would be required today within the well accepted deductivist position.

It is clear that we are not talking here about enumerative inductions, and it is argued that Newton's investigations about his Method of Fluxions are based on a specific type of analogies, that could be identified as *inductive analogies*. Newton's inductions can't be interpreted just as simple generalizations of individual facts without intervention of anything else in the process. *Analogies contribute to "soften" the inductive leaps.*

Newton's approach of inductive arguments requires on the part of the practitioner a cognitive model which could account for the specific use of the mathematical tools that are needed to carry out a justification of them. This model, based on creative analogies, surpassed the mere blind use of the mathematical techniques, as was posed above.

S. Barker (1989) presents the concept of *inductive analogy* as a kind of induction different from inductive generalizations in that the conclusion is a singular proposition. Enumerative inductions are concerned with the generalization of a result observed in a sample to a larger population from which the sample is taken. Conclusions from analogical inductions "express an empirical conjecture going beyond the available evidence".

Nevertheless, the term "induction" is used by Newton in a rather different way, as was pointed out before, and his inductive analogies could be reduced to the analysis of particular situations.

What is important in Newton's Fluxion Calculus is that the analogies used for the sake of argument are extracted from the relation between Geometry and Mechanics. The support for the geometrical arguments rely on elements from the domain of Nature - as he conceives the natural sciences- which imply that they must be submitted to observation and experience, although not necessarily extend it to more than one *good* case.

In order to conclude with the characterization of induction which Newton is involved, it can be said that it stresses the fact that there are properties that could be extrapolated from one initial case and applied to other or others, and that attempted to stand the importance of *agreement*, detected in the weighted evidence more than their frequency. The detection of these patterns of similarity links the inductive form of reasoning with the analogical inferences, for the latter gives ground to bet on this research trend, which is in some sense perceived and recognized in the evidence, or maybe we can say with Newton that it is "deduced from the phenomena", and the former allows to infer inductively from the similarities, acquiring knowledge from the interrelations among the specific cases by study.

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