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ABDUCTION AND CONJECTURING IN MATHEMATICS

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ABSTRACT

The logic of *discovering* and that of *justifying* have been a permanent source of debate in mathematics, because of their different and apparently contradictory features within the processes of production of mathematical sentences. In fact, a fundamental unity appears as soon as one investigates deeply the phenomenology of conjecturing and proving using concrete examples. In this paper it is shown that *abduction*, in the sense of Peirce, is an essential unifying activity, ruling such phenomena. Abduction is the major ingredient in a theoretical model suitable for describing the transition from the conjecturing to the proving phase. In the paper such a model is introduced and worked out to test Lakatos' machinery of proofs and refutations from a new point of view. Abduction and its categorical counterpart, *adjunction,* allow to explain within a unifying framework most of the phenomenology of conjectures and proofs, encompassing also the method of Greek *analysis-synthesis*.

0. Introduction

The contrast between the logic of *discovering* and that of *justifying* has been a permanent source of debate in the entire history of mathematics. In fact, there is a vast literature on the subject: it includes various contributions (by mathematicians, philosophers, psychologists, didacticians, etc.) embracing more than two thousand years of studies, from the pre-Euclidean mathematics up to nowadays. For surveys from different points of view, see: Cellucci (1998), Hanna (1989) and (1996), Lolli (1996), Rav (1999), Thurston (1994), Tymoczko (1985).

In the literature, there is a continuum of positions from those who have underlined the central role of formal proof in mathematics, insofar as it means something positively different from the heuristic used in the process of research, to those who have stressed the latter more, denying the value of the former in some cases. Moreover, many papers, written by people of different cultural areas in different times (from Pappus or Plato to Poincaré, Polya or Lakatos, through Descartes or Arnaud and Nicole), underline that in solving a problem, in conjecturing a hypothesis, in proving (or disproving) a result, a crucial point consists in the dialectic between an explorative, groping phase and an organising strategy which converges towards some piece of validated knowledge.

In this paper we show that *abduction*, in the sense described in Peirce's *Logic* (Peirce (1960), Vol.II, Book III, Chap.5, pp.372-388) plays an essential role in this dialectic: abduction reveals to be an essential resolutive move, after which the conjectures are formulated and allows the transition to the proving modality, which remains in any case deeply intertwined with it. This is partially in conformity with Peirce's claim that of the three logic operations, namely deduction, induction, abduction (or hypothesis), the last is the only one which introduces any new idea (Peirce (1960), 5.171).

Our thesis is based on a theorethical analysis and empirical observations made on subjects who solve problems in order to conjecture and to prove theorems. It has been elaborated analysing data collected: (i) from about 60 (high school and college) students, involved in a teaching experiment of elementary geometry since three years (within different environments: paper and pencil, computers); (ii) from the performances of experts (mathematics teachers in high schools and at the University), who solve elementary but not trivial problems and have accepted to speak aloud, while solving the problems; (iii) from the (rare) papers of professional mathematicians who have written about their processes of thought, while discovering new results.

The paper is divided into three chapters. In the first, some major features of the epistemological debate concerning the dialectic *discovering Vs/justifying* is summarised; in particular the topic *analysis Vs/ synthesis*. No originality is in this part, whose aim is to put forward the framework within which our researches are embedded. In the second chapter, we sketch our research on the role of abduction in mathematics; namely, we show that abduction is a resolutive move in the dialectic conjecturing/proving. In fact, it is crucial in producing conjectures; we will sketch a theoretical model suitable for describing the transition from the conjecturing to the proving phase: the model does not only fit with empirical data of our protocols, but will be used to give a fresh analysis of the well known example by Lakatos (1976), concerning the Euler conjecture on the edges, faces and vertices of a polyhedron. In the third chapter, the theoretical model is attacked from the point of view of categorical logic. In fact, the way abduction is used in our model hides an interesting phenomenon, namely the so called *adjunction*, which can be revealed using such logical tools. Abduction and adjunction are deeply connected; hence it seems to us that the above dichotomies (logic of discovery Vs/ the logic of justifying, conjecturing Vs/ proving etc.) are like the two sides of the same coin and that underpinning their contra position may be an ill way of posing problems. What we find is a common root, which has both cognitive and logical features, namely abduction and adjunction; the above contra position appears only if one remains at the surface of phenomena. Our result is still stronger, since analogous conclusions can be found also investigating the Greek analysis and synthesis with logical tools, as will be outlined at the end of the paper.

1. The historical & epistemological framework of the dialectic conjecturing-proving.

The creative and intuitive aspects of mathematicians' activity vs. the most rigorous ones have been scrutinised with different tools and ideas in the course of centuries, particularly concentrating on the relationships between logic(s) and mathematics conceived both as a product and as a process. In this order of ideas, roughly speaking, we can see different streams of thought (see Feferman, 1978, and Cellucci, 1998):

a. Many scholars have distinguished between a scientific logic and a natural logic: Descartes, Frege, Peirce, Dedekind.

b. Some have argued in favour of a scientific and formal logic which captures the essence of mathematics, namely its justificative aspects as well as its creative features: Aristotle (partially), Leibniz, Hilbert, Gentzen, Hintikka;

c. Some have seen the formal logic only as a justificative tool, claiming that intuitive and creative aspects of mathematicians' work elude a logical scaffolding and generally leave them to psychology: Frege, Feferman;

d. Some have tried to investigate the natural logic, possibly as a distinct 'discipline' from formal logic; they have investigated both the origin of mathematical ideas (Dedekind) as well as the features of mathematical discovering: Plato, Descartes, Peirce, Polya, Lakatos, Hintikka and Remes.

We furnish here only some examples to give the flavour of what is meant. For Frege, the formal logic is his *Begriffschrift*, that is the science which studies the laws of correct inference, whilst the natural logic concerns the ways after which inference concretely is performed and as such pertains more to psychology than to logic and is based upon empirical principles and not upon necessary and universal rules. (Frege (1969), *Logik*, p.4 and *Grundgesetze der Arithmetik*, vol.1, p. XIV). The relationship between the acknowledgement of the truth, which is a thought, as such not purely formal, and the development of the proving process, is very complex. Frege, in a letter to Hilbert in 1895 (Frege (1976), p.58) uses the metaphor of lignification: the development of a proof requires that it be built by the truth acknowledgement, as the development of a tree requires that it be soft and juicy in those points where it lives and grows. But the inference must become something mechanical to develop strongly, as a tree to become high must lignify its juices. In this sense, Frege develops the ideas of Leibniz about a *characteristica universalis* and a *calculus ratiocinator* (as he explicitly says in the *Introduction* to his *Begriffschrift*). The difference between Frege and Leibniz consists mainly in the fact that for Leibniz the formal logic concerns also the discovery of new results, a problem that Frege leaves to psychology: for Leibniz logic is useful "not only for judging what is proposed but also for discovering what is hidden" (Leibniz (1965), VII, p.523, letter to G.Wagner, 1696). The formal axiomatic method is a mechanical substitute of thought, insofar it "discharges imagination". In this sense Leibniz is similar to Hilbert, for whom the problem of discovering concerns logic and not psychology (Hilbert (1926), p.170).

Many scholars point out that Descartes illustrates the needs of a new logic of discovery, which cannot be embodied any longer in the formal logic of Scholastics: the Aristotelian logicians "cannot skilfully form a syllogism, which entails the truth, if they have not previously had its matter, that is, if they have not already known in advance that very truth which is deduced in it" (Descartes (1998), p.47. *Regulae ad directionem ingenii*, Regula X). The Aristotelian logic is "useless for investigating the truth of things, but it can only be useful for exposing to the others the reasons which are already known, hence it must be shifted from philosophy to rhetoric" (Ibid.). To ascend to the top of human knowledge people need a new logic (Regula II). The new logic has roots different from Aristotle and Euclid; in fact it goes back to Pappus and Diophantus (Regula IV), namely to the so called analytic method.

The same roots, namely the analytic method of Pappus, are invoked by many people who are put under our item d. For ex., the last chapter of Lakatos *Dissertation* at Cambridge (1956-1961) is devoted to the method of analysis-synthesis, as well as an address at a Conference in Finland (1973), in reply to a paper of Hintikka on the subject (all together, they constitute chapter 5 of Lakatos, 1978). Lakatos uses Pappus' and Proclos' definition of analysis to describe the process of discovering in mathematics, in particular that of criticising proofs and improving conjectures (Lakatos (1976), p. 9 and 75). The method, that Lakatos calls of thought-experiment or quasi-experiment from Szabó (1958), consists in decomposing "the original conjecture into sub conjectures or lemmas, thus embedding it in a possible quite distant body of knowledge" (*ibid.*). Polya in Polya (1990) and (1954) rephrases Pappus' method (he was called a second Pappus by Hintikka and Remes). Instead of a method upon which to build criticism to the way mathematical truths are presented in books after Euclid, namely with "finality-certainty requirements (which) survive in mathematics until today as the requirement of necessary and sufficient conditions" (from Lakatos (1978), p. 75), for Polya analysis is an auxiliary method, which helps in building up the rigorous proof: it helps in generating "a better understanding of the mental operations which typically are useful to solve problems" (*ibid.*), and as such it is studied as a useful pedagogical tool.

The most widely known formulation of the so-called method of *analysis-synthesis*, at least in mathematics, is the introduction to the Book 7 of the *Collection* by Pappus of Alexandria (Pappus, 1986). According to Pappus, "analysis is the path from what one is seeking, as if it were established, by way of its consequences (*ακολουθων*), to something that is established by synthesis. That is to say, in analysis we assume what is sought as if it has been achieved, and look for the thing from which it follows, and again what comes before that, until we get to something that is already known, or that occupies the rank of first principle. (...) In synthesis, by reversal, we assume what was obtained last in the analysis to have been achieved already, and, setting now in natural order, as precedents, what before were following, and fitting them to each other, we attain the end of the construction of what was sought."(*ibid.*) This text poses several problems of interpretation, concerning the *direction* of analysis-synthesis, as it is well known (for a wider discussion see Pappus, 1986). The second part of the passage seems to exclude that analysis is a downstream process (drawing logical conclusion from the desired theorem). On the other hand, if analysis is an upstream process (looking for the premises from which the conclusions can be drawn), then the synthesis is not the reverse of analysis, and it is useless. The problem is how can these two different concepts coexist in Pappus. The current interpretation is that they could coexist if we assumed that Pappus thought that all steps of deduction are convertible, that is the normal situation in geometry. By means of analysis we come to something already known. Then what is sought will be known if, through the synthesis, we test the reversibility of each step.

Hintikka and Remes propose another interpretation of the text, based on a different translation of the term *ακολουθον*. They suggest that "*το ακολουθον*... does not mean a logical consequence, but is a much more vague term for whatever 'corresponds to', or better, 'goes together with'" (Hintikka & Remes, 1974)). Hence they translate 'concomitant' instead of 'consequence'. Outlining the central role of auxiliary constructions in Greek geometry, they say that "the very purpose of analysis is to find the desired construction which is executed in the synthesis (...) If analysis is a series of steps which start from those parts of the figure which illustrate the desired theorem, and which establish connections between these and certain pre-existing entities, we of course do not obtain a synthesis in the sense of construction by simply reversing the order of these steps"(*ibid.*). The distinction between analysis and synthesis is then no longer a difference in direction. Also Descartes' methodological description of his algebraic method of analysis seems to agree with this interpretation. Instead of seeking a deductive connection between what is done and what is sought, he suggests to look for the dependencies between known and unknown quantities.

In this interpretation, called *configurational* in contrast with the directional one, the two ways, analysis and synthesis, could be really different and would represent two dual approaches to mathematical truths, the one complementary to the other. It is interesting to observe that these two ways have been interpreted by Scandinavian scholars (Hintikka, Remes, Mäenpää) within the machinery of Natural deduction (as formulated by Genzen, Prawitz, Martin-Löf) in a very suggestive way, which will result one of the major ingredients in our investigation.

2. Abduction

We shall show that abduction is a resolutive move in the dialectic conjecturing-proving. First, we will discuss how it enters into the theoretical framework of chapter 1; then we will give a model (based upon our empirical observations) which describes the way conjectures are produced by experts and how they manage the transition from the conjecturing to the proving phase. Afterwards we will use it to give a fresh and detailed analysis of Lakatos model of proofs and refutations.

The main points which emerge from the discussion in chapter 1 are:

1. There are two complementary ways of producing mathematical sentences and theorems, e.g. the logic of discovering or that of justifying, analysis or synthesis.
2. The two aspects seem to be always present in the activity of the working mathematicians and can be distinguished only for the benefit of a theoretical analysis. Hence mathematics as a human product must be explained not only taking both of them into account, but a model is needed where the two are integrated within a framework of continuity and complementarity, both from an epistemological and from a cognitive point of view.
3. Instead of a directional analysis of the process of research, it is more suitable to look at processes of conjecturing-proving within a configurational framework.
4. As a consequence, it is crucial the analysis of the interactions between the two above opposite streams. It is necessary to give a theoretical reason why such apparently contradictory approaches can live together in the real lif of mathematicians. A fresh analysis is needed, insofar as a lot of existing research seems to underline more the contradictions and the disconnections between the two: see for some different examples Tymoczko (1985), Davis & Hersh (1981) or Cellucci (1998).

We shall elaborate point 4 above, by describing a model which exhibits a fine dynamic structure of the two streams within a framework, which stresses more the continuity and the configurational aspects. This model has an experimental basis, but has been elaborated also through a theoretical analysis of problems under items 1, 2, 3 above. The interested reader can find more details in Arzarello et al. (1998a) and (1998b).

We shall expose the major features, shown by experts while solving elementary (but not routine) geometrical problems. Typically, a geometrical situation to explore is given, with some very open question to answer (e.g.: *under which hypothesis, such and such a property concerning the drawn figure does hold? make some conjecture and prove it*).

The subjects show successively two main modalities of acting, namely: exploring/selecting a conjecture and concatenating sentences logically. In fact, any process of exploration-conjecturing-proving is featured by a complex switching from the one modality to the other and back, which requires a high flexibility in tuning oneself to the right one.

Our aim is to analyse carefully how the transition from the one modality to the other happens. What a typical (clever) solver does, can be sketched as follows.

Phase 1

An exploring modality starts, with the use of some heuristic to guess what happens working on some particular examples, hence selecting a conjecture. The conjecture in reality is a working hypothesis to be checked: generally its form is far from a conditional statement and to confirm it new explorations are made by using some heuristic. Slowly, the solver detaches from the exploration process; generally, the situation is described by the subject in a language which has a logical flavour, but perhaps it is not phrased in a conditional form (if...then) nor it is crystallised in a logical form; on the contrary, the subject expresses his hypothesis not yet as a deductive sentence, but as an *abduction*, namely a sort of reverse deduction. In fact, generally the subject sees *what rule it is the case of*, to use Peirce language. Namely, she/he selects the piece of her/his knowledge she/he believes to be right; the conditional form is virtually present: its ingredients are all alive, but their relationships are still reversed, with respect to the conditional form; the direction after which the subject sees the things is still in the stream of the exploration: the control of the meaning is *ascending* (we use this term as in Saada-Robert, 1989, and Gallo, 1994).

Phase 2

A switching to the deductive form happens, because of the abduction. Now the control is *descending* and we have an exploration of the situation, where things are looked at in the opposite way: not in order to have hints for getting a conjecture, but in order to see in them why the regularity got with the abduction does work. The reversed way of looking at figures leads the subject to formulate the conjecture in the conditional form. Now the modality is typically that of a logical concatenation.

Phase 3

Now suitable, possibly fresh, heuristic is used, in order to prove the conjecture. Here the descending control is crucial: it allows the detached subject to interpret in the 'right' way what is happening, namely to produce logical concatenations. First they have a local character, then they are organised in a more global and articulated way. Here the detachment has increased: the subject has become a true *rational agent* (see Balacheff, 1982), who controls the products of the whole exploring and conjecturing process from a higher level, selects from this point of view those statements which are meaningful for the very process of proving and rules possible new explorations. In this last phase, conjectures are possibly reformulated in order to combine better logical concatenations and new explorations are possibly made to test them.

We observe that the exploration and selection modality is a constant in the whole conjecturing and proving processes; what changes is the different attitude of the subject towards her/his explorations and the consequent type of control with respect to what is happening in the given setting. It is the different control to change the relationships among the geometrical objects, both in the way they are 'drawn' and in the way they are 'seen'. This is essential for producing meaningful arguments and proofs. Also detachment changes with respect to control: there are two types of detachment. The first one is very local and marks the switching from ascending to descending control through the production of conjectures formulated as conditional statements (that is local logical concatenations) because of some abduction. The second one is more global and we used the metaphor of the rational agent to describe it: in fact it is embedded in a fully descending control, produces new (local) explorations and possibly proofs (that is global logical combinations). The transition from the ascending to the descending control is promoted by abduction, which puts on the table all the ingredients of the conditional statements: it is the detachment of the subject to reverse the stream of thought from the abductive to the deductive (i.e. conditional) form, but this can happen because an abduction has been produced. The consequences of this transition are a deductive modality and the new relationships among the geometrical objects of the figures. The inverse transition from descending to ascending control is more 'natural': in fact as soon as a new exploration starts again, control may change and become ascending again, even if at a more local level (with the rational agent who still controls the global situation in a descending way).

In short, the model points out an essential continuity of thought which rules the successful transition from the conjecturing phase to the proving one, through exploration and suitable heuristic, ruled by the ascending/descending control stream. The most delicate cognitive point is the process of abduction, crucial for switching the modality of control; one further relevant aspect is the change in the mutual relationships among geometrical objects, which are the essential product of such a switching. A further observation: as far as analysis-synthesis methods are concerned, our model is more in accordance with the configurational than with the directional approach; in fact abduction is active insofar as there is no pre-determined direction, the useful known and unknown quantities and their mutual dependencies are not completely fixed in advance (to rephrase Descartes), but must be searched for through an open exploration.

Let us now apply our model to the Lakatos method of *proofs and refutations* (see Lakatos, 1976) concerning the Euler formula V-E+F=2 on the number of vertices, edges and faces of a polyhedron.

Lakatos does not analyse the conjecturing phase: his work starts immediately after a first conjecture has been done. As he says: "The phase of conjecturing and testing in the case V-E+F=2 is discussed in Polya (1954). Polya stopped here, and does not deal with the phase of proving....Our discussion starts where Polya stops." (Lakatos, 1976, p.7).

It is a pity that Lakatos, as well as Polya, discusses only half of the story; in fact, the analysis of the two sides reveals strong elements of continuity; in the game of counter-examples of Lakatos, there are lots of abductions, which mark for example the switching from a global counter-example to new definitions (a global counter example produces a criticism of the conjecture, not of the proof, according to Lakatos). In fact such abductions allow to bound and better refine the domain of validity of a proposition: it is the case when the counter-example shows that the proposition is "in principle true, but admits exceptions in certain cases"(*ibid.*, p.24).

But if abduction is a main ingredient also in the Lakatos logic of discovery, there is a big difference with the example of the conjecturing phase, which illustrates the different modalities of the refinement of conjectures discussed in Lakatos. The main point is the general control of the rational agent with respect to the situation: whilst in our model there is a positive logic (namely, something is first conjectured, then proved), in Lakatos, so to say, there is a logic of *not*: abduction does happen within this new frame, dual in a certain sense to the one seen before: we shall discuss in chapter 3 this duality in a very precise manner.

Let us explain the differences in a more explicit way. In our model (producing a conjecture and proving it) we can distinguish:
(i) A context, more precisely a fragment of a theory of reference, let us say P, within which explorations and conjectures are drawn (e.g., some piece of elementary geometry);
(ii) A surprising or interesting situation, let us say E, worthwhile to be explained by some conjecture, namely by the (a) reason why E holds within P (as an example, think to the surprising situation of the square built on the diagonal of another square in Plato's *Meno*, which has a double area of the starting one). We can represent the resulting problematic situation by the following diagram (it does not necessarily mean that one is operating within a formal system, but only that one is looking for a reasonable hypothesis for E within a certain mathematical domain of discourse P, see Rav (1999), p.11):

$$P \vdash (\ ?\ ) \rightarrow E;$$

(iii) Dynamic exploration (like in *Meno*), with ascending control, allows the subject to find such an hypothesis P', as a 'possible cause' of E within that context: namely P' is produced with an abduction, and then starts the descending control which produces possibly the final proof in the end, within a logic of discovering/proving, which result so deeply intertwined:

$$P \vdash P' \rightarrow E.$$

The example of Lakatos and consequently his theory of proofs and refutations, which consists in refining an existing conjecture and proving

it, can be pictured in the following way:
(i) At the beginning, there is a conjecture, namely some sentence E which holds possibly within a context P:

$$P \vdash E;$$

(ii) Because of some thought experiment, a surprising or interesting situation appears, namely a counterexample **c**, which has a global character (see above):

$$P \vdash \neg E(\mathbf{c});$$

Hence the problematic situation now consists in explaining the reason of the counterexample, that is of which rule this counterexample is the case within the given context; hence by a new abduction, we get such a reason, let us say P^-

$$P \vdash P^-(c);$$

(iii) Now, a new resolutive move starts, typical of what we have called the *logic of not*. The new move consists in investigating the connections between the cause of the counterexample, namely P^-, and the conjecture $P \vdash E$. In fact, P^- is possibly a reason why the conjecture does not hold; hence it is reasonable to look within the context P for some new hypothesis, let us say P^+, which eliminates P^- and consequently, the counterexample. Lakatos' paper describes how the subject can find such a P^+, within the context P, through a dynamic exploration: namely how P^+ is produced with a new abduction, of the type described in the preceding example:

$$P \vdash P^+ \rightarrow \neg P^-.$$

Generally, with the new hypothesis P^+, one has all the ingredients for producing a proof of the conjecture E, within the enriched theory $P\&P^+$:

$$P \,\&\, P^+ \vdash E.$$

The process of proving consists in combining logically the pieces of information and local logical connections produced in the previous phases; the control in this last phase typically switches from an ascending to a descending modality.

It is also interesting to observe that such dynamics are described by Lakatos himself in Lakatos, 1978, p. 93 and followings, where he describes the machinery of proofs and refutations within the framework of Pappusian analysis-synthesis: "The analysis provides the hidden assumptions needed for the synthesis. The analysis contains the creative innovation, the synthesis is a routine task for a schoolboy.....However, the hidden lemmas are false. ....But nevertheless we can extricate from the analysis (or from the synthesis) a 'proof-generated theorem' by incorporating the conditions articulated in the lemmas." (*ibid.*, p.95).

As a last comment, the two types of searching hypotheses that we have illustrated, can be analysed and evaluated also from the point of view of economicity, as Peirce does in his discussion of abduction.

3. The Theoretical Model from the Point of View of Natural Deduction.

The discussion in chapter 2 shows essentially two ways of attacking problems:

$$P \vdash (\,?\,) \rightarrow E \qquad [1]$$

$$P \,\&\, (\,?\,) \rightarrow E \qquad [2]$$

In both cases abduction plays an essential role in reversing the course of thought (from ascending to descending control); in the second case the use of counterexamples seems to be at the origin of a more involved course of thought, which we have called the *logic of not*.

In this chapter, we will show that the two modalities are dual, in a precise technical sense. First, let us approach the question in an intuitive way; later on, we shall give a few technicalities. The two problems [1] and [2] have analogies and differences: namely, in both cases one is looking for some missing (or hidden) hypothesis; but in the first, the context is fixed and the hypothesis is searched for within that context, whilst in the second, it is the context, that is the (implicit or explicit) domain of discourse, to be challenged because of the counterexample. In fact, the concrete examples we have observed in our experimentations and the example of Lakatos corroborate this observation. In the former, which are non-routine problems, no challenge is made to the (implicit) theory and the new hypothesis can be found within the theory (a typical trivial example could be: "*under the hypothesis that the two sums of the opposite sides of a quadrilateral are equal, a circle can be inscribed into the quadrilateral*"); in the latter, it is the implicit theory to be challenged and a new hypothesis within it that must be formulated (a typical example being theorems concerning trapezia, where the hypothesis of the convexity of figures must be explicited; a similar case is precisely the example discussed by Lakatos).

Natural Deduction (see Prawitz, 1971) precisely illustrates this phenomenon, provided one takes into account an approach to logic which shows also the dynamic aspects of thought, like the categorical approach to Natural Deduction (see Martin-Löf, 1984). In such an approach the above relationship between conjunction and implication (now we are working within a formal system, for example in the way described in Prawitz, 1971):

$$P \,\&\, Q \vdash E \qquad\qquad P \vdash Q \rightarrow E$$

are described in term of a fundamental concept in category theory, namely the so called *adjunction* (see Crole, 1993). We can say that the implication is the right adjoint of conjunction.

Hence, adjunction illustrates from the logic point of view (provided you approach logic according to category theory) a phenomenon which has a cognitive counterpart in abduction.

To be more precise, solving such problems like

$$P \,\&\, \_ \vdash E \qquad P \vdash \_ \rightarrow E \qquad [3]$$

requires finding two dual relationships $\varphi(\_)$ and $\psi(\_)$ (two adjoined functors in the language of categories); to do that, one must go backwards, from the conjecture E towards the unknown hypothesis _, with a thinking style which is 'opposite' to the deductive one and that has more a configurational than a directional pattern. The backwards direction, which is mirrored at a cognitive level by what we have called *ascending control*, has been emphasised also by Lakatos: "Whether a deductive system is Euclidean or quasi-empirical is decided by the pattern of truth value flow in the system. The system is Euclidean if the characteristic flow is the transmission of truth from the set of axioms 'downwards' to the rest of the system -logic here is an *organon of proof*; it is quasi-empirical if the characteristic flow is retransmission of falsity from the false basic statements 'upwards' towards the 'hypothesis' -logic here is an *organon of criticism*." (Lakatos, 1978, p. 29).

Backwards direction and the logic of not, typical of finding solutions to the $\varphi(\_)$ in [3], correspond to what Lakatos calls retransmission of falsity; it is however questionable that their nature implies quasi-empiricism in mathematics.

On the contrary, the relationships between abduction and adjunction which exist behind the solution processes to equations [3] are similar to the relationships between informal and formal proofs. The distinction has been made by many scholars (see Kreisel, 1970, pp. 445-467): the former are those "of customary mathematical discourse, having an irreducible semantic content" (Rav, 1999, p.11) and the latter are syntactic objects of a formal system. In fact, abduction and backwards strategies seem to exhibit essentially two typologies (corresponding to [1] and [2], respectively), whose psychological and epistemological features, essentially embodied in abduction, have been described in chapter 2. The formalisation within the systems of Natural Deduction and the categorical interpretation given in chapter 3, essentially based upon adjunction, seem to represent the formal counterpart of such processes. Of course it is questionable that adjunction incorporates everything, as well as it is perhaps debatable that every informal proof has a formal counterpart (as the so called *Hilbert's Thesis* says, see Rav, 1999, p.11); something similar happens when one compare the intuitive notion of effectively computable function with the technical definition of partial recursive function (*Church's Thesis* says that the technical definition captures all of the intuitive concept). We do not suggest here to invent a new Thesis, to support the adequacy between the two notions; what we underline is that like in the previous two examples (computable functions, proofs), the existence of an intuitive notion has deepened its analysis through mathematical investigations without claiming a new epistemology, in the same manner many things made by people who make conjectures and proofs can be explained using suitable mathematical (and traditional) tools of analysis.

We wish to conclude the paper with two more observations. First, the works of Hintikka and Remes (1974) and Mäenpää (1997) have used deductive logic to interpret the configurational aspects of analysis; they used essentially the systems of Natural Deduction with (in the last papers) the machinery of types of Martin-Löf, see Mäenpää (1997), that is in the version which shows more connections with computer science. They showed that "the Greek method of geometrical analysis can be generalised into a method of solving all kinds of mathematical problems in type theory by taking into account inductively defined problems, which are characteristic of programming. The method known as top-down programming turns out to be a special case of analysis." (Mäenpää, 1997, p.226). It is intriguing (and corroborating the ideas above) that we find also from another approach a reduction to the same formal machinery!

Second -as a further corroboration- the same strategy (of looking for suitable economic hypothesis in a backward approach) that we have analysed above is at the basis of many programs for automatic theorem proving in elementary geometry, e.g., those which use the Gröbner-Buchberger algorithm, see Chou (1988): the algebraic varieties, which are the geometric counterpart of the algebraic symbolism incorporated into the software, are the natural model of a dynamic logic, where the method of resolution means looking for a new economic hypothesis, which allows to restrict the domain of validity of a supposed conjecture, in order that this really becomes true, see Cox et al. (1992), pp.280-296. The strategy is very similar to that illustrated in problem [2], modulo the algebraic translation.

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