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HOW INFINITIES CAUSE PROBLEMS IN CLASSICAL PHYSICAL THEORIES

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Introduction

Although it may sound as some sort of confession, I must warn the reader that I am a strict finitist. I hold the position that mathematics can do (extremely) well without the notion of infinity, whether actual or potential. As to the technical feasibility of such an undertaking, I must refer the reader to Van Bendegem [1987], [1992] and [1993a]. As it happens, this is not the subject of this paper, it is therefore not important whether the idea of strict finitism is ludicrous or not. The subject that concerns me here is how well "the other side" is doing. Let me clarify this last statement. Surely it must be an additional argument in support of strict finitism, if it can be shown that within the classical infinitary theories themselves these infinities give rise to rather serious problems. As I already indicated, I will not discuss this problem from within mathematics, but instead I will concencrate my efforts on one of mathematics most beautiful and most successful applications: physics. In particular, classical Newtonian mechanics is the main subject I want to deal with. The aim of this paper is to show that infinities cause all sorts of bizarre problems in the framework of classical mechanics. The core problem is the loss of determinism. To be quite cleár about the matter, what I do not show is that *all* infinities should be eliminated in order to restore determinism. Actually, I will mostly deal with infinities "in the large" and not "in the small". It is thus not a defense of strict finitism (in certain parts of physics), as said, it just adds some other arguments in its favour.

On the conceptual level, I will use and rely on a theoretical device that probably most philosophers have declared obsolete: supertasks[1]. To be quite honest, until the publication of a paper by Victor Allis and Teunis Koetsier [1991], I shared this attitude. However, having read the Allis-Koetsier paper, I was quite astonished to see that (apparently) the last word had not been said or written about supertasks. In fact, this paper will add some new ideas to the subject. Thus, it can just as well be read as the "re-opening" of a classified subject.

A quite general argument

The basic equation of classical mechanics is, of course, the Newtonian law $F = m.a$ or $F = m.d^2x/dt^2$ or, if x is expressed as a function of t, $F = m.d^2x(t)/dt^2$. As we all know, classical mechanics is considered to be the determinist theory par excellence[2]. How the determinism is derived is easy enough to show in the simple case of a single equation. Suppose that F and m are given and that there are two solutions x(t) and y(t). Thus we have that:

$$m.d^2y(t)/dt^2 = F = m.d^2x(t)/dt^2. \qquad (*)$$

Hence, it follows that

$$d^2y(t)/dt^2 = d^2x(t)/dt^2.$$

Integrating two times, produces the final result that:

$$y(t) = x(t) + c1.t + c2,$$

where c1 and c2 are two constants to be determined by the initial data. Suppose that these data are (i) $x(0) = y(0)$ and (ii) $dx/dt(0) = dy/dt(0)$. Then it follows straight away that both c1 and c2 equal 0. Hence, the solution is unique. Perhaps all this is considered to be entirely trivial. And perhaps it is, but, nevertheless, let me go through the argument once again. However, this time I will make explicit in a somewhat Lakatosian fashion, all the hidden assumptions.

We start with $F = m.d^2x(t)/dt^2$ and $F = m.d^2y(t)/dt^2$. If we set the right-hand sides of both equations equal to one another, then it is as-

[1] The classic treatise on the subject is Grünbaum [1968] to be completed with Grünbaum [1973], pp. 630-645. See also Salmon [1970] and Ray [1991], pp. 14-23.

[2] I do emphasize the "considered to be" as I will show in the sequel of this paper that this view is not correct on a standard "orthodox" reading. The same point is made in Earman [1986], chapter III.

sumed implicitly that F is not infinite. Thus, (C1): no infinite forces are allowed. If F is not infinite, we find (*). Assuming that m is not infinite, we can divide both sides by m. Thus, (C2): no infinite masses are allowed. If the first integration of both sides is to have any sense, then it must be excluded that infinite accelerations are allowed. Thus, (C3): no infinite accelerations are allowed. Finally, to have no problems with the second integration, no infinite velocities are allowed. (C4): no infinite velocities are allowed. Altogether, we have uncovered four hidden assumptions each one excluding a type of infinity to occur. Of course, this argument on its own does not imply that if one of the conditions (C1)-(C4) is violated, then determinism does not hold. In fact, this is the purpose of this paper: to show that this is indeed the case. Unless one is willing to give up determinism, the occurrences of these infinities must be avoided at all times. Also, it does not imply that the four conditions are independent of one another. And, finally, it does not imply that masses, forces, accelerations and velocities have finite upper bounds. It may very well be that arbitrarily large values are allowed, only excluding the occurrence of the infinite value itself[3].

Let me consider a first example to clarify the problem we are dealing with. Suppose that the trajectory of a particle P obeys the following equation between a starting time t_o and an end time t_f. Suppose that the interval $[t_o,t_f]$ is split up in an infinite series of intervals $T_i = [t_i,t_{i+1}]$ such that $t_{i+1} - t_i = (t_o - t_f)/2^{i+1}$. Suppose finally that the movement x(t) of the particle P during interval T_i is:

$$x(t) = \sin(2\pi(t-t_i)/(t_{i+1}-t_i)) \qquad \text{for } t_i \leq t \leq t_{i+1}.$$

In geometrical terms, this corresponds to a particle following a sinusoidal

[3] In Allis & Koetsier [1991], an ingenious supertask version is presented of a perhaps not so well-known paradox, Ross' paradox. Imagine an empty urn and an infinite number of labeled balls (although this is not necessary, it facilitates the presentation of the argument). One minute is divided in an infinite number of decreasing intervals in the usual manner. In the first interval, balls 1 up to 10 are put in the urn and 1 is taken out. At the n-th interval balls 10.(n-1)+1 up to 10.n are put in the urn and n is taken out. After one minute, the urn must be empty, because in the n-th interval, n was taken out, for all n. On the other hand, at each interval 10 - 1 balls were added to the urn, so it should be filled with an infinite number of balls. To show that Ross' paradox is an impossible supertask, I had to invoke the condition that there is a largest finite speed (not necessarily c). It is however an open question whether this condition is really necessary. See my [1993b] for the full details.

path such that the wavelength becomes infinitely short. If we now want to find out where the particle is at time t_f, the answer must be that we cannot tell. For, obviously, the function x(t) taken over the whole interval $[t_o,t_f]$ does not have a limit. Since the mass of the particle does not enter into the discussion, it is obvious that condition (C2) is trivially satisfied. So is condition (C4) as it is equally obvious that we can calculate the first derivative at any moment t. However, if we calculate the acceleration, we note that the acceleration is unbounded. In other words, before we reach t_f, it must have become infinite. Thus condition (C3) is violated and, via the Newtonian equation, condition (C1) will be violated as well. What I have presented here in a rather abstract fashion corresponds perfectly well to the basic example of a supertask: the Thomson lamp.

Consider an eletrical circuit consisting of a power source, a switch and a lamp. Splitting up the one-minute time interval in the above fashion, one performs the following task. During T_0 the switch is on and the lamp is burning, during T_1 the switch is off and the lamp is off, during T_2 the switch is back on, etc. The question is quite similar: what will be the state of the lamp (on or off) after one minute? The only difference is that the function x(t) describing the movement of the switch is a step-function rather than a sinus function[4].

The fact that classical determinism is so easily threathened by such simple cases as the Thomson lamp probably inspired Christopher Ray when he wrote in his [1991, p.19]:

> "Perhaps the moral of this tale is that, when we conjure up empirical fictions, we should not be too surprised when our stories end unhappily, even if they do have impeccable mathematical credentials."

However, the moral being told, the problem is how to deal with these empirical fictions.

[4] The only reason I prefer the presentation in terms of sinus function, is that one avoids the counterargument that the Thomson lamp is impossible because in a step-function the first-order derivative, i.e. the velocity, is not defined at the corners where the step-function changes value. This is an entirely irrelevant detail of the presentation.

Supertasks old and new

How one is to solve a supertask-problem is not a straightforward matter. Generally speaking, two strategies can be considered. The first, (S1), acknowledges that to accept the result of the supertask implies thereby the loss of determinism. However, if one does not wish to give up determinism, then one will try to show that the supertask is impossible. Hence the problem cannot occur. The second strategy, (S2), accepts indeterminism as such. The supertask is a process allowed for by the theory, it follows that there is no reason to reject it. Putting it slightly differently, the two strategies come down to this. Let T represent classical mechanics. Let Σ represent the set of all (partial) models of T. Among these models one finds the Thomson lamp. Hence, it is impossible that there could be a derivation in T that T is deterministic. Strategy (S2) comes down to accepting this situation. The first strategy (S1) proposes to limit down the set of models Σ to a set Σ^* such that T restricted to that set does become deterministic, i.e. it is provable that T is deterministic. This still leaves open two sub-strategies: either one puts restrictions on the models themselves or one introduces in T additional principles that are only satisfied in Σ^*. This paper follows the second sub-strategy: at least the principles (C1)-(C4) must be added to classical mechanics to safe-guard determinism.

Let me therefore show, first of all, that indeed the four principles are needed. The Thomson lamp is an example of the need for (C3) and, by implication (as masses do not enter into the problem), for (C1). As for principle (C2) — as far as I know, little or not attention has been paid to this case — the following example may show its need:

The infinite mass balance problem

Suppose we have a balance B with two scales. Split up a one minute interval in an infinite number of time intervals in the standard way, and perform the following procedure. We have at our disposal an infinite number of masses m_i, such that $m_0 = m$, $m_1 = m/2$, and, generally, $m_i = m/i$. The total sum of the masses is $\Sigma\, m_i = \Sigma\, m/i = m.\ \Sigma\, 1/i$. As the second factor is the harmonic series, the total mass is infinite. What we simply do is to add mass m_i to the left scale at interval T_i and mass m_{i-1} to the right scale at the same interval. Thus, at T_0 m goes left and nothing goes right. At T_1, m/2 is added to the left scale and m to the right scale. The question to be answered is this: after one minute, what is the position of the scales of the balance?

Two answers are equally possible. The first is to say that at every stage, the left scale of the balance contains more mass than the right scale, thus at all times before the end of the minute, the balance is out of its equilibrium position. As nothing is done during the whole procedure to compensate for the non-equilibrium position, the balance will remain so at the end of the minute. However, one can argue that at the end of the minute, for each mass m_i in the left scale, there is a corresponding mass m_i in the right scale, so both scales contain exactly the same mass, therefore the balance must be in equilibrium.

Variations on this theme are easily found. Suppose that we are at the end of the preceeding supertask and suppose the balance is in equilibrium. Take away any mass m_i you like from, say, the right scale. This cannot change the balance's equilibrium, since $m - m_i$ is still infinite. Repeat the procedure as many times as you like, the balance will remain in equilibrium. In fact, even if one takes away all the masses $m_{2i} = m/2i$, the balance remains in equilibrium. However, the sum of all masses taken away, $\Sigma\ m/2i = m/2\ .\ \Sigma\ 1/i$ = infinite. Hence the obvious need for a principle like (C2).

It must however be noticed that (C2) only addresses the problem of infinite masses. What is not at issue here, is the acceptability of the idea that masses can be split up in an infinite number of parts (much as time and space intervals). In other words, the balance problem is not avoided by claiming that mass is not infinitely divisible[5]. Although at first I did hold this claim, two beautiful examples found by Tom Etter[6] showed me

[5] This statement must be interpreted within the framework of classical mechanics. The masses m_i can, if necessary, be represented as point masses, thus their extension is not an essential feature. Hence, we can define a function m: D → R such that D is a denumerable set of place locations, x_i, and such that $m(x_i) = m_i$. Considerations about the size of elementary particles do not belong to this theoretical framework.

[6] Tom Etter - the editor of *ANPA-West*, the (American) journal of the ANPA, the Alternative Natural Philosophy Association - presented me with these examples in private correspondence more or less as problems for strict finitism. In the latest issue of *ANPA-*

to be mistaken.

The case of the moving bar

Suppose we have two bars A_1 and A_2 parallel to one another separated by a distance d. In the space between, a series of vertical bars is located, according to the figure below.

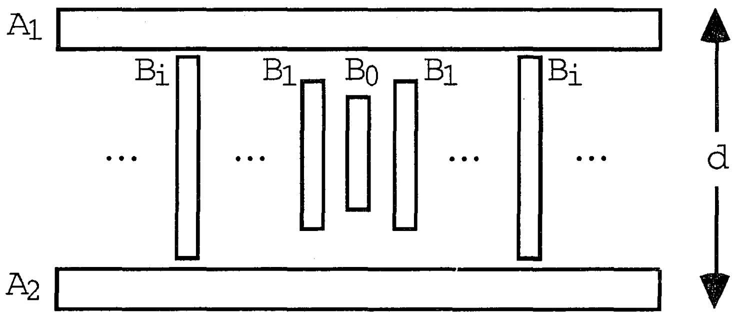

Figure 1

The height of the middle bar B_0 is d/2, of the next bar B_1 is d/2 + d/4, generally the height of bar B_i is the sum of all distances $d/2^{j+1}$, for $0 \leq j \leq i$. Thus no bar B_i touches either A_1 or A_2. The question to be answered is this: suppose we move the top bar A_1 in the direction of A_2. What will happen? Obviously A_2 must change its position as a result of the movement of A_1. If not, the distance between A_1 and A_2 would decrease to 0, but that is impossible because of the presence of the vertical bars. Therefore A_2 must move. But the only way this can be achieved is if A_2 is touched by at least one pair of vertical bars B_i. In its turn the movement of the bars B_i must result from a push of A_1. However, if A_1 makes contact with B_i, it must have made contact with B_{i+1} before, as B_{i+1} is closer to A_1 than B_i. In general, A_1 can touch B_j only if it is has touched B_{j+1} before that. As this series has no first element, apparently A_1 can touch no vertical bar at all. Thus the movement of A_2 is impos-

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West, see Etter [1992], he presented the Zeno bowling game with the "challenge" to calculate the final outcome of the collision process (as it is described in this paper).

sible. Given this result, we must conclude that either A_1 cannot be moved at all or, if it is moved at all, it touches no pair of vertical bars. And this is odd to say the least.

The Zeno bowling game

Suppose that on a line an infinite series of masses B_i is laid out, such that no two masses are touching, and such that both the masses and the volumes from left to right become smaller (see figure below).

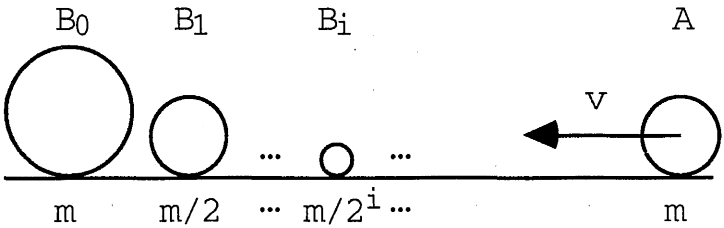

Figure 2

Thus $B_o = m$, $B_1 = m/2$, $B_2 = m/2^2$, and generally, $B_i = m/2^i$. Thus the total mass of all B_i's is finite and equal to 2m. If the volumes decrease sufficiently fast, then all masses can be aligned within a finite stretch of the line (though this is not strictly necessary; the argument goes through with point masses). Finally, suppose that all these masses are in rest. From the right, still on the same line, a mass A is approaching the series of B_i's at velocity v. What will happen? If we reason along similar lines as in the first example, then it seems that nothing will happen. If A collides with, say, B_i, it must have collided with B_{i+1} before that. Thus A cannot collide with any B_i. But as A has a constant velocity, this would mean that A moves "through the B_i's" (whatever that is supposed to mean). Again an odd conclusion to say the least.

The solution to both these problems is easily seen if the second example is slightly transformed into a far more familiar case. Take the Zeno bowling game. Instead of keeping the B_i's separate, join them. Instead of spheres, turn them into rectangles with the same height H for each one, but with diminishing length L_i. Let the sum of all these lengths be L. Then it is obvious what we obtain: a nice rectangle of size L by H with mass 2m (see figure 3).

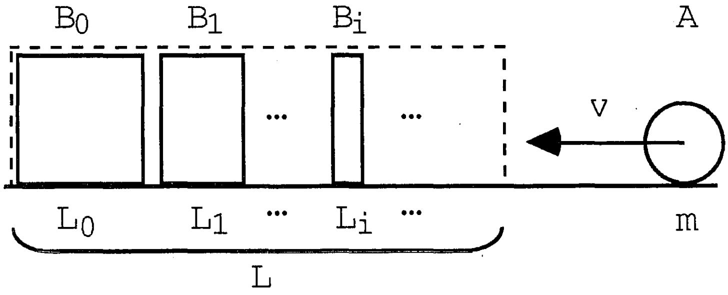

Figure 3

The only thing we have done is to split up the rectangle in an artificial way such as to create an infinite series. However if A collides with the rectangle (as a whole) there will be no problem. This is a perfectly classical case, perfectly solvable and as deterministic as we may wish the world to be.

To put it slightly differently, when we are talking about an object — whether it is thought of as one solid whole, or whether it is thought of as sliced up in an infinite set of parts — normally speaking, we include the boundary or limit points. In the case of the series of B_i's it is easy to see that the boundary (whatever it is) does not belong to the B_i's themselves. That is to say that the first thing A gets into touch with is not one of the B_i's but one of the boundary or limit points. As from that moment onwards the collision process has started, there is no problem to accept that at a certain time t_i A will collide with B_i. Although this solves the problem, we must resist to try and picture the process. As in Zeno's arrow, it does traverse all the distances, an infinite number of them, in a finite time. Our imagination falls short in visualing this process, but theoretically and mathematically there is nothing wrong at all. The solution with the boundary points applies equally well to the bar problem. Join all the bars together and look at the closure of this figure. In fact, there will be boundary points that are precisely a distance d apart. This means that the structure consisting of the union and closure of all vertical bars is already touching (at least in some points) the horizontal bars. Thus, as soon as A_1

is moved, so is B, and, therefore, so is A_2.[7]

Note too that these examples illustrate in a quite clear way that not all infinities in classical mechanics are to be eliminated. Limit points are typically products of an infinite process. If one were to insist on the elimination of all infinities, then the proposed solution would be exluded and both the bar and the bowling game would remain problematic cases to solve.

Returning to the main argument of this chapter, let me illustrate the need for (C4). The example presented here is a slightly adapted version of one of the cases treated by Allis and Koetsier in their [1991]. Consider two locations P_1 and P_2, separated by a fixed distance d. Split up the time-interval in the standard fashion. At the beginning a mass m is placed at P_1. During T_0 it is transferred to P_2, back to P_1 during T_1, back to P_2 during T_2, and so on. The question is: where will m be at the end of the minute? The argument will sound familiar by now. If we write out the function x(t) that represents the movement of m, then we will find an oscillating function with no obvious limit. Hence m can be either in P_1 or in P_2. Indeterminism once more. But note that the velocity must increase beyond any finite limit. For, in interval T_i a distance d must be covered, thus the velocity is $v_i = d/T_i = d/(t_{i+1}-t_i) = d/(1/2^{i+1}) = 2^{i+1}.d$. Hence (C3) is violated and the process is excluded[8].

Summarizing, what these few examples clearly show is that (C1)-(C4) really are necessary to avoid the conclusion of indeterminism. However, it does not show that the subset of models that satisfies these

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[7] Nevertheless it must be mentioned that reasonings of this type must be very carefully dealt with. Consider the following example. On a line are two masses in rest both having mass m/2 and separated by a distance d. From the right a mass m is approaching with velocity v. The first collision give the right most of the two m/2 masses a velocity 4v/3 and mass m a velocity of v/3. The right mass m/2 collides with the left mass m/2 and passes on its velocity, such that the end situation is the following: mass m has velocity v/3, the middle one is at rest and the left one has velocity 4v/3. As long as there is a separation between the two m/2 masses, this is the answer to the problem. However, if d = 0, i.e. when the two masses touch, the solution jumps discontinuously to a quite different solution: at the end of the process, mass m from the right has velocity 0, whereas m/2 + m/2 = m gets velocity v. Is this discontinuity something to worry about or not?

[8] This example is well-known in the literature on supertasks. It is a variant of a Black's transferring machine Beta. See Grünbaum [1968] for a full discussion of these machines.

four principles, will coincide with the deterministic models. In other words, what guarantee do we have that indeed all these models are determinist?

Will four be enough?

Although I have not been able — in fact, I am quite skeptical about the fact that it could be possible at all — to find a formal proof that (C1)-(C4) are sufficient, I did find additional arguments in its favour. If one looks through the literature on (in)determinism in classical mechanics, two typical cases appear over and over again: collisions involving more than two masses and gravitational singularities not involving collisions.

Collisions involving more than two masses

It is generally accepted that simultaneous collisions of three and more masses are underdetermined. The story is well-known[9]. If we have three masses m1, m2 and m3 with known velocities v1, v2 and v3 simultaneously colliding, then, apart from a set of special cases, an infinity of solutions is possible describing the situation after the collision. After all, all we have are the equations of conservation of angular momentum, momentum, and energy. There are more unknown elements than independent equations, hence we have underdetermination. But, on top of that, there is the curious result that one of the most obvious solutions to this problem does not solve the problem at all.

It seems quite natural to invoke what I will call a limit principle (LP). A simultaneous collision is then considered as the limit of a series of non-simultaneous collisions, where the limit is taken over the time-interval between collisions. Thus instead of the single simultaneous collision, one considers, say, m_1 colliding with m_2, m_2 then colliding with m_3, perhaps m_3, after that, colliding with m_1, and so on. As each of these two-collisions is perfectly solvable, the whole process is. Hence we obtain a unique solution in every case. Thus the limit is uniquely determined and, in fact, one does find a limit value that corresponds to the

[9] See, e.g., Penrose [1989], p. 168 and Stewart [1989], p. 40. It is worth mentioning that the earliest formulation of this problem, known to me, is Gale [1952], where the failure of the limit principle is shown.

three-collision. But, alas, if the limit is taken from a different direction — we consider the collision of m_1 and m_3 first and then all the other collisions as they occur — we do find a limit value but it turns out to be different from the first one. Hence, the hope for a unique solution for the three-collision problem seems pretty thin. Does this then not show clearly that even (C1)-(C4) do not suffice to safe-guard determinism? Yes, but on one condition only, namely that (LP) is consistent with (C1)-(C4).

Although the details of the argument can be found in Van Bendegem and Cornelis [1993], let me present its main features. Perhaps it is not obvious at first, but (LP) does carry a rather important hidden assumption. As the three-collision problem is seen as the limit of a sequence of two-collisions, it is assumed that as long as we do not reach the limit, the process described is that of a sequence of *consecutive* two-collisions. It is vital to the argument that no pair of two-collisions overlap. If it did, then the limit procedure does not apply, for overlapping two-collisions pose exactly the same problems as three-collisions. In the equations the three masses will participate and underdetermination reappears. The only situation where (LP) does apply is then where it is assumed that the two-collisions are instantaneous. But, instantaneous collisions violate (C3).

This is easily seen with a very simple example. Take two masses of equal magnitude, $m_1 = m_2$. m_1 is at rest and m_2 approaches m_1 with velocity v. The equations tell us that after the collision m_2 is at rest and m_1 has velocity v. Thus, m_1 has changed its velocity from 0 to v. As this can only happen during the collision and if the collision is instantaneous, this requires an infinite acceleration, in violation with (C3). Hence (LP) is inconsistent with (C3). Of course, it still remains to be shown that if (C1)-(C4) are accepted and if, hence, (LP) is rejected, then a single solution is found for the three-collision. But this is actually the case.

An important remark to make is that this analysis shows that (at least) the process of a collision requires a minimum time interval Δt. This does not imply unfortunately — for the strict finitist, that is — that time in classical Newtonian mechanics should be considered discrete. After all, if that were the case, integrals and derivatives would loose their standard meaning. It does imply, however, that physical processes require a minimum time-interval. Putting it differently, it does seem to be an argument in favour of an ontology of time-intervals rather than an ontology of time-points. Physically speaking, intervals are more plausible entities, whereas mathematically speaking, time-points are needed. But, as time-points can be derived from time-intervals (in a classical mathematical framework), it opens the possibility to consider a basic ontology of time-intervals for classical mechanics[10]. Admittedly, this single case does not allow us to generalize without any restrictions. Perhaps in other cases, time-points are more appropriate. Nevertheless, it does support the idea (in favour of strict finitism) that the basic language of physics is a discrete language, at least as far as time is concerned.

Gravitational singularities without collisions

If classical gravitational theory is included in classical mechanics, processes can be found that involve singularities — hence the loss of determinism — without collisions. Fortunately, most of the "hard" work has been done by others. Thus, it has been proved that[11]:

(a) (Painlevé): If a physical system is given consisting of n bodies and the gravitational forces between them, then, if a singularity occurs, at least two bodies will occupy the same position in the limit. In other words, if there is to be a singularity, at least two bodies involved in the process will approach one another arbitrarily close. This, however, does not tell us, whether they will occupy the same position within a finite time. If so, we have a collision singularity, if not, a non-collision singularity (assuming here, of course, that masses are taken to be point masses).

(b) (Painlevé): If only three bodies are involved, all singularities are collision singularities. Thus for that case at least nothing new happens that has not been discussed yet. What about four or more?

(c) (Von Zeipel): If a non-collision singularity occurs, then at least one of the bodies involved must escape to infinity in a finite time. This is rather good news for our purpose here. It means that the escaping body must reach an infinite velocity in a finite time, but that is precisely in violation with (C4).

One might remark that the problem would be a rather uninteresting one if non-collision singularities did not occur at all. However, what has

[10] Setting aside of course the difficult question of reformulating the basic physical principles of Newtonian mechanics in terms of time-intervals instead of time-points. It seems a rather safe option to assume that the result will not be elegant.

[11] The results that follow are neatly summarized in Xia [1992].

been shown many times over and recently by Zhihong Xia in his [1992] is that they do occur. In fact, Xia explicitly presents an example using just five bodies. Rather intriguingly, the case four is still open.

In conclusion, assuming that only singularities pose a threat to determinism, the conditions (C1)-(C4) do exclude the two sole possibilities — collision and near-collision processes — to generate singularities. Hence, it is justified to claim that the finitist approach (that is, restricted to the acceptance of (C1)-(C4)) does manage to safe-guard determinism. Determinism, that is, in classical Newtonian mechanics involving gravitation or not. But where does this leave us with respect to other physical theories?

What about other physical theories?

Although any treatment in depth would require to go through all known physical theories, this attempt must be put aside for two reasons: my lack of knowledge of many parts of physics and lack of space (even if the first restriction did not apply.) I will therefore focus my attention on two special cases: special relativity and electricity theory. One might wonder about the latter. Should not general relativity (GRT) be a more interesting candidate for investigation than electricity theory? There are at least two good arguments to support this idea. First, the black hole discussion is basically a discussion about singularities. Physical laws break down at or inside black holes. Some reject the existence[12] of naked singularities — invoking "cosmic censorship", see Penrose [1989], corresponding nicely to strategy (S2) — whereas some do accept them. Secondly, and this is perhaps somewhat less well known, there is another type of indetermi-

[12] Quite recently, John Barrow in his [1992], esp. p. 187, has shown that care must be taken as to the interpretation of "existence" here. If one reads existence in a constructive fashion, then some singularity theorems (proving the existence of a singularity under certain conditions) must be rejected as they rely on *reductio ad absurdum*. This opens a road of inquiry not taken up in this paper: is it possible to interpret (C1)-(C4) as a kind of constructivistic principles?

nism inside general relativity generated by the so-called hole argument[13].

As is well-known, the basic equations of GRT formulate a connection between the energy tensor T and the metric tensor g. In the best of cases, T determines g uniquely but in most cases, different solutions are possible up to a diffeomorphism d. In other words, consider a model M of GRT, and represent this model as M = <m, g, T>, where m is a manifold, g the metric tensor (describing the metric structure of m) and T is the energy-tensor. Then there exists at least one diffeomorphism d such that M' = <d*m, d*g, d*T> and such that d*m = m and d*T = T, yet d*g ≠ g. Hence, M' = <m, d*g, T> is also a solution of the basic equation. Thus if a energy-tensor is given, the space-time structure is not uniquely determined. This result on its own need not be a threat to determinism in the standard sense (as I have been using the term up to now). But, as it turns out, it is possible to find a diffeomorphism d, such that for all time moments t before a given time t_0, d is nothing else but identity, and for some later time t' in some region of space-time, d is different from identity. In other words, up to t, the sequence of events is perfectly determined, yet, somewhere in the future — i.e., in the region where d differs from identity — two sequences of events are possible. This is indeterminism as clear as one can have it.

This is deep result and a difficult one to solve for that matter. However, as is clear from the discussion about the hole argument (see Earman [1989]), the problem is not related to infinities at all and definitely not to principles such as (C1)-(C4). Hence, little is to be found here for the strict finitist, except the non-trivial observation that the exclusion of infinities will not be the last word on the problem of determinism and indeterminism in the physical world (as described in GRT). All this being said, let me now turn to special relativity.

Infinities in special relativity

At first sight, one is tempted to think that problems, if any, one could encounter in special relativity theory (SRT) concerning infinities, should be less numerous. The obvious reason why is that at least one of the

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[13] An excellent overview of the hole argument is Earman [1989], esp. chapter 9. See also Norton [1989]. The argument is not a novel one; as a matter of fact, Albert Einstein himself already formulated the argument in 1913-14.

conditions (C1)-(C4) is trivially satisfied, namely (C4), since all velocities v must satisfy $v \leq c$, where c has a finite value. This is unmistakenly true. It is equally obvious that (C2) becomes more important. Now the mass m of a body moving with velocity v is $m = m_0/(1 - v^2/c^2)^{1/2}$, where m_0 is the rest mass. Thus as v approaches c arbitrarily close, m will tend to infinity. It is therefore not necessary to start out with an infinite mass (or an infinite number of finite masses) to obtain an indeterministic answer. In SRT one might just as well start with a finite mass m and let it move at velocities closer and closer to c. It must be clear by now that most of the problems I talked about in the framework of classical mechanics can be repeated here if appropriately reformulated in terms of SRT.[14] Hence I will skip this problem and focus instead on a new type of problem that, I believe, is typical for SRT.

It is a Thomson lamp, it is not a Thomson lamp

Suppose we have two reference frames X and X'. In X there is a Thomson lamp at rest. Now suppose that X' is moving with velocity v relative to X. Applying the Lorentz transformations, it is straightforward to calculate that, if the Thomson lamp needs a total finite time Δt in X, say one minute, then the time measured in X' will equal $\Delta t/(1 - v^2/c^2)^{1/2}$. Thus if v approaches c, this time-interval will become infinite. But if the time-interval becomes infinite, then there is no problem with the Thomson lamp, as seen from X', since we only have a problem if the supertask is executed in a finite time. But this leads to the remarkable conclusion that an impossibility in one framework — the Thomson lamp in X — is a possibility seen from another framework — a quite ordinary lamp that is switched on and off an infinite number of times requiring an infinite (more precisely, an unbounded) time. Or, to put it in even stronger words, in X the observer will conclude that determinism is at risk, whereas the observer X' will conclude that there is no problem at all. Apparently, determinism is dependent on the chosen framework. A

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[14] A distinction should be made between the collision problems that deal with instantaneous action only and the gravitational problems that deal with action at a distance. According to many authors the latter case does not make much sense within the framework of SRT. The appropriate background theory, it is argued, for these problems is GRT.

strange conclusion indeed!

One might object to this example that, although v can tend to infinity, in every specific frame X', v will have a finite value and, hence, in X' too, the lamp will reach its end-point, thus turning it into a Thomson lamp. This objection can be rejected by slightly adapting the example. Imagine that in X we have a Thomson lamp such that the time-intervals are of the form $1/2^i$, the whole process taking exactly one minute. Now imagine that the velocity of X' is related to the time-intervals in X, in such a way that the velocity of X' at interval $1/2^i$ is

$$v_i = c.(1 - 1/2^{2i-2})^{1/2}.$$

Thus I assume that at each interval the speed of X' is raised by a given amount. Although the underlying mechanism to realize such a situation might prove impossible, there is nothing wrong with the supposition itself. Much as in the twin paradox it is assumed that changes in velocities, i.e., accelerations, can be instantaneous, the same is accepted here. It is now straightforward to calculate that the time-interval $1/2^i$ measured in X' comes out exactly 1/2. Thus in X' what will be observed is an ordinary lamp that goes on and off forever in an extremely regular fashion.

Of course, the solution to the problem is easy enough. Either one reasons in a classical way within the reference frame X. There is definitely nothing wrong to use the argument that has been proposed for the Thomson lamp in a classical setting. Or one might invoke principle (C3) and ask how the "boosting" of framework X' is supposed to take place. As it is supposed to be instantaneous, it violates (C3). If, however, a minimum time is "reserved" for the acceleration, it is clear that X' cannot reach arbitrary velocities. Note that it is not necessary to use the argument that the coordination established between the time-intervals in X and the boosting process for X', is not physically realizable. After all, it need not be a pre-established coordination; perhaps highly improbable, but the process could have taken place by chance. The point of the story is that it is possible to have two frames X and X' such that two observers will report quite different events.

An important remark to mention is that the inverse process is not possible as time-intervals are always "stretched". That is, it is not possible to have at the same time (i) a given framework X wherein a process takes place that has no finite upper bound in time, and (ii) a framework X', moving relative to X, wherein the time measured for the process taking place in X is finite in X'. In other words, an "ordinary" lamp cannot be observed from any reference frame as a Thompson lamp.

Let me now deal with the last topic selected from a quite different area, namely electricity theory. The reason why I have chosen this topic is that, on the one hand, it is a rather unusual domain as a subject for philosophers of physics, and, on the other hand, it is the only example I know where infinities are explicitly considered useful for practical applications. As Zemanian [1991] puts it:

> "... questions, which are meaningless for finite [electrical] networks, crop up about infinite ones and lead to novel attributes, which often jar the habits of thought conditioned by finite networks." (p. vii)

Infinities in electricity

As might be expected, the questions mentioned in the above quotation have to do basically with the existence conditions for unique solutions of Kirchoff's laws supplemented with Ohm's law[15] (these play exactly the same role as the basic Newtonian equation):

> "A case in point is the occasional collapse of such fundamentals as Kirchoff's laws. Indeed, Kirchoff's current law need not hold at a node with an infinite number of incident branches, and his voltage law may fail around an infinite loop". (p. vii)

The problem that arises is unavoidably the problem of finding conditions that will select a unique solution. Zemanian notes the following:

> "There is a dichotomy in the theory of infinite networks, which arises from the fact that Ohm's law and Kirchoff's voltage and current laws do not by themselves determine a unique voltage-current regime except in certain trivial cases. This leads to two divergent

[15] Kirchoff's current law states that at any given node in an eletrical network, the algebraic sum of all branch currents flowing through this node is zero. Kirchoff's voltage law states that, given a directed loop in an eletrical network, the algebraic sum of all branch voltages in the loop is zero. Ohm's law is the law that voltage = resistance x current.

> ways of studying infinite electrical networks. One way is to impose those laws alone and to examine the whole class of different voltage-current regimes that the network can have. ... The second way, ..., is motivated by the following question. What additional conditions must be added to Ohm's law and Kirchoff's laws to ensure a unique voltage-current regime?" (pp. ix-x)

The first strategy corresponds perfectly to strategy (S1) to accept indeterminism and to study all solutions. The second strategy, quite similarly, corresponds to the approach (S2) outlined in this paper: what principles do we have to add to restore determinism? In the case of electrical networks, imposing restrictions that eliminate infinities does some, but not all of the work:

> "One conspicuous requirement that suggests itself is finiteness of the total dissipated power. This suffices for some networks but not for all. In general, what is occurring at infinity has to be specified as well. ... in [other] cases it is necessary to know what is connected to the network at infinity if a unique voltage-current regime is to be obtained." (p. x)

Perhaps this last statement might sound rather enigmatic. A simple example may illustrate what is meant. Suppose we have a network of the following form (see figure 4, numerical values have no importance here). We now have two possibilities to "complete" the network at infinity. Either the two branches remain open and hence we assume that the circuit at infinity is open. In that case we have an open-ended circuit at infinity. Or we can assume that there is short circuit at infinity. As it turns out, the one assumption does not give the same result as the other. In other words, although the voltage-current regime is uniquely determined one a choice is made, it is not if the situation at infinity is not specified.

It follows from these observations that, if there is to be any hope to obtain a unique solution for the voltage-current regime, then the network should be insensitive to what happens at infinity. Although this condition is strictly speaking not of the type (C1)-(C4), it can be reformulated as such.[16] After all, what is says is that numerical values of some relevant elements of the network at infinity do not play any part in the determination of a unique solution. Hence if there is a principle that excludes actual infinite values for these elements, then these values can be dismissed and should not enter into the calculation. Such a principle is very much of the form (C1)-(C4).

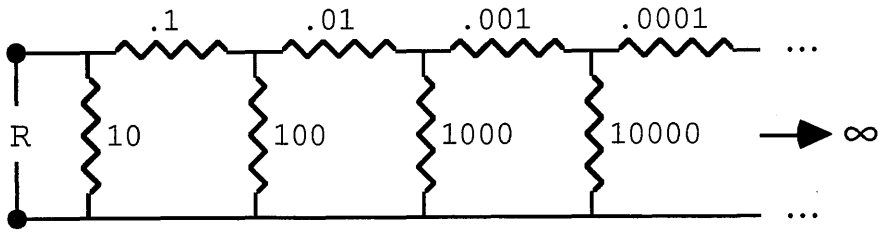

Figure 4

Although these few considerations have barely scratched the surface of this rather exciting theory, let me reiterate what I said in the beginning of this chapter: my aim was utterly modest. I did not try to solve any deep problems, rather it was my intention to wet the reader's appetite for these strange problems that crop up when infinities are allowed. If the reader is convinced or willing to agree that the supertask-problem and the related problem of infinities is not confined to classical mechanics but goes well beyond the limits of that theory — actually, that it pervades the whole of physics — then I have reached my goal.

Conclusion

There is, of course, one major gap in the subject. What about quantum mechanics? Surely, this must be a topic worth considering. I have, however, refrained from tackling this difficult and treacherous subject because

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[16] Perhaps it may sound somewhat trivial, but (C1)-(C4) are evidently not applicable in their present form. Mass is not a relevant datum, velocities have to be reinterpreted as currents, and so on. This being said, there is no doubt that it can be done.

the very problem of determinism versus indeterminism is not even settled yet, perhaps, one might argue, not even clearly understood. It then seems rather premature to consider imposing restrictions such as (C1)-(C4) (or analogical statements) in order to safe-guard determinism. This being said, one cannot help but being struck by the fact that the collapse of the wave function is an immediate event (if it is an "event", of course) requiring no time at all. Thus we do seem to have something comparable to infinitely fast change, if, of course, any genuine change is occurring at all. However, at this stage, I have no arguments that would help me to improve this weak analogy into a strong argument. I therefore leave it as an open problem.

I warned the reader at the beginning of this paper that I am a strict finitist. The reader should not deduce from this that I am a determinist on the basis of the arguments outlined in this paper. After all, the indeterminist can reason in the quite opposite way: Thomson lamps and similar devices show that classical mechanics is fundamentally indeterministic. My only purpose in this paper has been this: the discussion about determinism and indeterminism in classical physics is extremely closely tied to problems about infinities. Hence a reflexion about infinities — what is their status?; what is their nature?; do we need them or not as essential ingredients for our theories about nature? — is called for. What the outcome will be of this reflexion has not been my subject here. However, the option to leave infinities out altogether is not as silly as it may seem at first glance. It does solve some problems.

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