FALLIBLE INTUITIONS: THE APRIORI IN YOUR MATHEMATICS

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Introduction

In his: *The Nature of Mathematical Knowledge*,[1] Philip Kitcher argues that we do not have apriori knowledge of mathematical truths. While arguing that nothing warrants holding a mathematical claim with the certainty required for knowledge of it to be apriori, he reminds so-called apriorists that some mathematical knowledge is based on long and complex proofs. Kitcher has his apriorists suggest that we need not be certain of all our apriori knowledge. At this suggestion Kitcher warns:

> "One can easily go astray here, by conflating apriori knowledge with knowledge obtained by following a non-empirical process." p. 43

In reaction, Mark Steiner querried, in his review of Kitcher[2]:

> "But why is this going astray? We have here two notions of the a priori, of which the second is no less interesting and crucial than the first."

After offering an interpretation of Kitcher's contribution, this essay offers a tentative definition of this "second" notion of apriori as a "non-empirical process." Apriori knowledge, as Kitcher defined it, is not crucial for the mathematical knowledge of individual human beings. Nevertheless, what Kitcher showed is crucial and interesting for our understanding of mathematical knowledge. Kitcher showed that pursuit, let alone attainment, of absolute certainty is irrelevant to the origin and development of the body of mathematical knowledge possessed and used by the human race.

I have set primary and secondary goals for this essay. The primary goal is to present the hypothesis that philosophers of mathematics confront two categorically different levels mathematical knowledge. There is mathematical knowledge as part of the culture of various communities and of humanity as a whole. Let us label mathematical knowledge as a community's possession: *communal mathematical knowledge*. Terms such as 'Chinese mathematics' or 'Chinese mathematical knowledge' are available for reference to the communal knowledge of specific communities. Communal knowledge has its processes of origin and growth in the various communities as well as in the whole human community.

What kind of thing is communal mathematical knowledge? A book entitled "The Nature of Mathematical Knowledge" should tell us something about it even if it does not give a full characterization of its nature. Indeed Kitcher's book gives us important information about communal mathematical knowledge. He tells us, amongst other things, that we can understand its origin and growth with the concepts sufficient for understanding the origin and growth of communal empirical scientific knowledge. In particular we do not need to use the concept of apriori knowledge. We can also develop ideas of his book to distinguish the nature of communal mathematical knowledge from the nature of mathematical knowledge in individuals.

What is an appropriate label for the fragments of communal mathematical knowledge each of us has. Candidates were terms such as: 'personal','private,' 'individual,' and 'idiosyncratic.' The term should neither glorify nor denigrate mathematical knowledge at the individual level. Furthermore the term should not connote other philosophical views and problems. A grammatical phrase seemed suitably neutral. Let us label the mathematical knowledge of each individual the *first person singular* knowledge of that individual. For brevity, I shall frequently write merely '*SINGULAR*' for 'first person singular.' Choice of this grammatical label suggests relabelling communal knowledge with 'first person plural' or 'plural.' Nevertheless, I shall retain the label 'communal.' I want the label to suggest that a community's mathematical knowledge is a single social reality; not merely bits of information scattered throughout the plurality of its individuals.

In each of us there are processes of origin and growth of our individual first person singular mathematical knowledge. How are these processes of origin and growth at the singular level related to processes of origin and growth at the plural or communal level? Part of my primary goal is to emphasize fundamental differences between knowledge acquisition processes at the singular level and corresponding processes at the communal level. My secondary goal is to remind you of how crucial for your singular mathematical activity and knowledge are the fallible insights attained by special ratiocinative non-empirical processes. At the singular level there are processes, properly characterized as apriori, which are crucial for the growth of individuals' mathematical knowledge even if no increases in the communal body of knowledge are accurately described as warranted by an apriori process.

How shall I move towards these goals? Part I, the major Part, is my case for distinguishing between communal and singular mathematical knowledge. I develop my definition of 'apriori' by altering Kitcher's[3]. So, Part II introduces a bit of his epistemology and his definition of 'apriori.' Part III offers a tentative definition of 'apriori' for first person singular knowledge.

Part I: *Communal distinguished from singular knowledge*

We should reinterpret Kitcher's thesis about the non-existence of apriori knowledge as a thesis about communal mathematical knowledge; not about first person singular knowledge. To justify necessary conditions for my reinterpretation, I will make a Wittgensteinian case that there is a distinction between communal and singular mathematical knowledge[4].

Can I consistently assert: No communal mathematical knowledge is apriori although some singular mathematical knowledge is apriori? Development of a consistency argument provides a warning against a philosophic temptation to reduce the communal to the singular. The foundation of my consistency argument is the semantical fact that many common nouns enjoy an ambiguity in the way we use them referentially. Or in medieval terms: We can use them, and their associated forms, with either simple or personal supposition.[5]

We use common nouns to refer to types or kinds of thing; this is to use them with simple supposition, *(suppositio simplex)*. We also use them to refer to individuals of the types; this is to use them with personal supposition, *(suppositio personalis)*. When we use them with personal supposition we can refer to some individuals of the type or all of the individuals of the type. When we use them to refer to all of the individuals of the type a medieval term for the usage was 'confused personal supposition,' *(suppositio personalis confusa.)* The term 'confused' did not label the usage as defective. In contemporary times 'fused' is a less confusing term for this use of 'confused.' It is perfectly legitimate and necessary to make assertions about all of the individuals of a type while not making an assertion about the type as a unit itself. Nevertheless, it is easy to confuse *supposi-*

tio personalis confusa with *suppositio simplex.*

Why is it easy to be confused between whether or not we are talking about the type or each and every member of the type? The useful and legitimate notion of set or class facilitates confusion. As something individual a set satisfies the need for a single unit as the referent in sentences properly analyzed as having simple supposition. However, as a complex of its members, while being in some way nothing more than its members, a set satisfies philosophic scruples, perhaps implicitly held, that we are talking of individuals. We are now at risk of equivocating between interpreting a referent of a common noun as a type and as a set. The risk of such type/set confusion is high if you hold, as I do, an "aristotelian" presupposition that the type exists in the individuals of the type. Our philosophic fears of reification make the risk even higher.

Of course, reference to definite finite subsets of individuals of a type are not likely to be confused with reference to the type. We also use common nouns to pick out a definite individual of the type or a definite proper subset of the individuals of the type; this is to use them with *suppositio personalis determinata.*

For examples, consider the common noun 'people' and cognate terms. Let us start from the most determinate.

In 'That person is intoxicated' and 'The people in Columbus, Ohio on Sept. 1, 1988 are South of Lake Erie' the term 'people' is used with determinate personal supposition; we use it to stand for a definite person or set of persons and, in principle, we could pick out each person in the subset.

In 'All people are mammals' or 'Every person is a mammal' the term is used with confused personal supposition. The reference is personal in so far as we are asserting that each individual person is a mammal. However, the reference is properly labelled 'confused.' We "fuse" together all the people who were, are, and will be into a collection and make an assertion about each via reference to this collection. This fusion is necessary for the reference because we have no other access to each of the persons we want to talk about other than by gathering them into a collection with the universal quantifier.

In 'People have evolved from non-human primates' we are talking about the human race. We are not asserting that every person has evolved from some non-human primate, from a non-human species, a non-human population, or anything else. If we use 'people' with personal supposition we can truly assert: 'No person has evolved' or 'No people have evolved.' So, although I do not recommend using it except to make a philosophical point, there is a consistent reading of: 'People have evolved although no people have evolved!'

Philosophic fear of reification tempts us to think that if the human species is anything at all it is the set of all people who are living or who have ever lived. If we succumb to that temptation, there is a further temptation to make sense of the odd, if not nonsensical 'The set of people has evolved.' by reading it as: 'Every person has evolved.' We should resist such temptations.

Consider the common noun 'mathematical knowledge.' There is a consistent reading of: 'Mathematical knowledge is not apriori although my mathematical knowledge is apriori!' How? 'Mathematical knowledge' abbreviates a more complex term such as: 'human mathematical knowledge' or 'mathematical knowledge of people.' When its use is not too awkward let us use one of the more complex, but more precise, terms. What is the supposition of 'mathematical knowledge of people' when we assert 'The mathematical knowledge of people is apriori?' A tempting answer is: Fused personal supposition. The temptation is nearly irresistible if we presuppose that the referent (supposition) of 'mathematical knowledge of people' is the set of all the bits of mathematical knowledge had at some time or other by some person or other. With fused personal supposition it is appropriate to replace the definite article 'the' with the universal quantifier to get: 'All mathematical knowledge of people is apriori,' which we can articulate as: 'For each and every person each and every item of mathematical knowledge of that person is obtained by an apriori process.'

Using fused personal supposition, how do we read Kitcher's thesis: 'The mathematical knowledge of people is not apriori?' We can express it as the universal negative: 'No mathematical knowledge of people is apriori'? The universal negative can be articulated as: 'There is no person who knows anything mathematical by an apriori process.' Let us label this articulated universal negative 'the distributed reading of Kitcher's thesis.'

The distributed reading is inconsistent with what many people hold to be true. Kitcher's historical opponents such as Kant and Frege, I, and perhaps you hold: 'My mathematical knowledge is apriori.' (I personally qualify my claim by modifying 'mathematical knowledge with 'genuine' or 'real' to allow for the fact that I have to rely on authority for so much of what I use in mathematics.) In 'My mathematical knowledge is apriori,' the pronoun 'my' gives the common noun 'mathematical knowledge' determinate personal supposition. The supposits are our respective collections of singular mathematical knowledge.

We see, then, that if we use 'mathematical knowledge' with personal supposition on both occasions it is inconsistent to assert: 'Mathematical knowledge is not apriori although my mathematical knowledge is apriori.' The inconsistency can be resolved by using the first 'mathematical knowledge' with simple supposition. This resolution of the inconsistency is formal because it does not exhibit the simple supposit of 'mathematical knowledge.'

It is important to have shown that there is no formal inconsistency in accepting Kitcher's thesis while holding that my own best mathematical knowledge is apriori. As examples below remind us, we speak of mathematical knowledge as a single science different from the individuals in whom it originated and the billions of individuals who use it and develop it. Nevertheless many of us, fearful of reificiation, hold, explicitly or implicitly, that claims about mathematical knowledge have to be reduced to claims about what individuals know if we ever need to articulate them precisely.

What, though, is the simple supposit of 'mathematical knowledge' in 'Mathematical knowledge is not aprior?' Some reminders of how we speak indicate that 'communal' is a suitable label even if we do not know precisely what it is. There is a cognitive social institution with distinctive practices and assertions, with a history and social norms for its development. 'A science' and 'a body of knowledge' are also suitable terms for such a cognitive institution.

A distinction between communal religious belief and singular religious belief might clarify the distinction I am marking in mathematical knowledge. Catholic belief has an origin and twenty centuries of development. Some theologians assert of their faith as a communal heritage: 'Catholic belief is revealed by God.' My Catholic belief is a fragment of orthodox and heterodox claims from this heritage. Certainly the origin and history of my Catholic belief is not Church history. I can accurately assert that my Catholic belief has not been revealed to me by God. Of course, without individuals, with their spiritual biographies, in whom Catholic belief originated and develops there would be no Catholic belief. Communal mathematical knowledge stands to mathematical knowledge in individuals as does Catholic belief to the beliefs in individual Catholics.

J.P. Van Bendegem makes a nice use of this distinction between mathematical knowledge as a body of knowledge and mathematical knowledge as knowledge in individuals. He made the distinction in his "Fermat's Last Theorem Seen as an Exercise in Evolutionary Epistemology,"[6] Professor Van Bendegem exhibits significant approaches in struggles with Fermat's last problem.

He does so to propose a preliminary and very general model for evolutionary accounts of the growth of mathematical knowledge in regard to this problem and associated areas. We can read: "It is obvious that the model presented here is not well enough articulated to justify the choice of a particular evolutionary model. Nothing here has been said about the carriers of the mathematical ideas, viz. the mathematicians themselves. The story presented here takes place in realm of mathematical problems and methods." p. 356.

The realm of mathematical problems and methods is that realm I call our communal knowledge. The realm of the carriers of mathematical ideas comprises what I call singular mathematical knowledge.

In a recent book combining psychology of mathematics, history of mathematics, and philosophy of mathematics there is a provocative use of the distinction.[7] One of Ernst Welti's major goals is to show that development of strict finitist mathematics marks progress in mathematical knowledge. His strategy employs two principles. One is "Das Prinzip der kognitiven Regression" which states:

> Der Wissenschaftliche Fortschritt ist oft mit einer kognitiven regression verbunden, d.h. er fuhrt vom entwicklungsmässig Spätern zum immer Früheren zuruck. p.14

Development of topology and group theory are offered as examples of scientific progress by making explicit mathematical ways of thinking appropriate to less developed stages of mathematical thinking than those reached in the age when topology and group theory were developed. How do we judge which ways of thinking are developmentally earlier? We are asked to make a judgment about the developmental level of mathematics in various cultures through history. We need, then, to make a judgment about development in mathematical knowledge as a social institution. How can we make a such judgment? Welti turns to mathematical knowledge at what I call the singular level.

Welti sketches Piaget's account, as elaborated by H. Wermus, of the development of mathematical ways of thinking in individuals' mathematical knowledge. To apply the individual development scale at the social level he introduced his other strategic principle: Psychohistorisches Grundprizip. It tells us:

> Die (kognitive) Ontogenese rekapituliert in vielen Fällen die Wissenschaftsgeschichte. p. 11

For the case of apriori knowledge, I disagree with this principle. I hold that at a certain stage individuals develop the capacity for causing themselves to gain mathematical knowledge by apriori processes. But this advance to apriori processes in individuals recapitulates no knowledge acquisition process in the history of mathematical science. Nevertheless, despite that disagreement with Welti, I agree with the distinction between the two developmental processes.

To elaborate on the difference between communal and singular knowledge let us observe five logical differences between 'I know p' and 'We know p.' As I use 'we' in the following observations, the term 'we' is not to be read as an implicit quantifier. The term 'we' is read as a quantifier, and not as a pronoun, if we read 'We know p' as giving some statistical information such as: All of us know p, Almost all of us know p, Many of us know p, etc.,.

1. 'We know p' does not imply 'I know p.' I can know that experts know p without knowing p myself. At most my saying 'We know p' implies that I should know that we know p. For instance, In a context of talking about completeness proofs for a relevance logic I might say that the axiom of choice is equivalent to Zorn's Lemma. Someone might ask me whether I know that for a fact. My honest reply would be that I do not "really" know it but I do know, by reliable hearsay, that it has been proved.

Mention of reliable hearsay is embarrassing. But we do use it. Reflect on how much you know of various subjects has "just been picked up" by you without your knowing when or how you came to know it.

2. 'I know p' does not imply 'We know p.' A mathematician who has just developed a proof for a hitherto unproved theorem may be entitled to say 'I know it.' However, that theorem is not yet known. There are social conditions which have to be met for an individual's knowledge to become part of collective knowledge, eg., publication and ratification by a community of experts. For instance, Fermat may have known his notorious "last theorem" when he made his marginal note. However, we cannot say that we know Fermat's last theorem or that it is known.

3. 'We know p' does not even imply 'There is someone who knows p.' It could happen that all who knew p are dead or have, at least, long forgotten p. It might even be that p is a corollary of items of mathematical knowledge q and r, which someone long dead once proved. The person who proved q and r long ago never even thought of corollary p. No one now bothers to relearn q and r. We simply claim we know p as a result of the known q and r which no individual now knows.

In Part II we will observe that the knowledge of both individuals and the community have to be produced by a warranting process. Are these the same kind of processes? My thesis on reading Kitcher is that the answer is NO. However, prior to my arguments for interpreting Kitcher we can see the logical independence of 'We know p by process X' and 'I know p by process X.

4. 'I know p by process X' does not imply 'We know p by process X.' This extends logical observation (2) above. I also add that we, as a culture, cannot do what I as an individual can do. For instance, there is no sense in which a community can inspect diagrams, as you and I can, to work through the proof in "Meno."

5. 'We know p by process X' does not imply 'There is someone who knows p by process X,' let alone that 'I know p by process X.' Item (5) is perhaps the only controversial logical observation.

Let X be the process by which a community, with the due thought and deliberation processes Y of some of its specialists, accepts an proposition p in its cultural heritage. Such a process is a reliable method for a community. Consider any one of the experts. A typical expert might vigorously deny knowing p by the process of careful acceptance of tradition. The proud expert might assert: 'I don't know p by tradition; I know p by proof.'

My thesis on Kitcher applies this logical independence to the case of mathematical knowledge. We now need a closer examination of Kitcher's epistemology.

Part II: *The apriori in Kitcher's psychologistic epistemology.*

Beware! I am sketching Kitcher's epistemology thoroughly convinced of my distinction between communal and singular knowledge; especially of those logical distinctions just observed. Such convictions may distort my presentation of his epistemology because I agree with so much of it. He marks a distinction between 'we know' and 'I know.' Indeed his Preface lead me to distinguish between communal and singular knowledge. I think Kitcher underestimates the importance of the singular and its independence from the communal. So, I may be "skewing" his entire epistemology to mark a sharper distinction than his between 'we know' and 'I know.' In his Preface he announces that his theory of mathematical knowledge differs from standard accounts in three ways. I begin my sketch with my interpretation of these three differences.

(i) Mathematical knowledge originates in sense experience

and is justified by sense experience.
(ii) It is primarily a community that has mathematical knowledge. Thus an account of how '*we* know,...,' is more fundamental than an account of how 'I know,...'
(iii) Because we focus on how a community comes to acquire and to extend its knowledge, we need to examine history to uncover the origin of the community's knowledge and the methods of the growth of the community's knowledge.

My difference with Kitcher is on [ii]. As I would put it: Neither communal nor singular knowledge is more fundamental for the philosophy of a science. I could also express my difference as: If philosophers of a science adopt a naturalistic epistemology, they must use both the sociology of knowledge and the psychology of knowledge without conflating one with the other.

In his first chapter, Kitcher argues for a psychologistic epistemology. Psychologistic epistemologists hold that true belief requires a special kind of cause to be knowledge. He points out that psychologistic epistemologists distinguish between natural processes which cause knowledge and those which do not. Those which cause knowledge are *warrants*. He explains why it is premature to analyze warrants at this stage in the development of cognitive science and epistemology. He pleads a need for more data of the type he develops in his book before we define 'warranting process.' Such data are historical accounts of how various bodies of knowledge originate and are rationally developed by people individually and as communities. I suspect that Kitcher is laying the groundwork for a sociologistic epistemology more than for a psychologistic epistemology.

Talk of rational development reveals that the causal processes which are also warrants have a normative aspect. The normative aspect guides Kitcher in challenging processes as warranting apriori knowledge. What is the normative aspect and how are warrants challenged? Communities in fact develop standards for awarding, or withholding, the status of knowledge to beliefs brought about by various processes. If epistemology is ever fully naturalized there will be a natural evolutionary explanation of how and why we have our epistemic standards. Until then, psychologistic epistemologists challenge belief causing processes with the epistemic standards of their communities. With use of community standards, psychologistic and naturalistic epistemologists argue as traditional epistemologists. They appeal to our understanding of proper use of 'know.' Kitcher reminds us of four social conditions, which relative to our linguistic intuitions about 'know,' undermine our acceptance of a belief causing process as warranting knowledge. I present those conditions as challenges. The question, then, is how effectively they challenge both: 'We know' and 'I know.'

Four social challenges to belief causing processes

1. People attack the reliability of a belief causing processes.
2. People make a theoretical case against what is believed.
3. People of good epistemic reputation simply disagree with what is believed
4. People make a case that the way of speaking is obsolete.

Challenge #4 may be a bit puzzling. It undermines conceptual analyses using discarded definitions. Kitcher shows[8] how 'Acids contain oxygen' expressed conceptual knowledge at the beginning of the nineteenth century while failing to do so at the end. Change of chemical theory and corresponding changes in definitions made what 'Acids contain oxygen' expressed obsolete. So, in 1900 no one could responsibly teach chemistry, i.e., pass on to students what WE know of chemistry, by professing that acids contain oxygen.

The challenges begin with 'people' to emphasize the social character of the challenging conditions. However, 'people' has determinate personal supposition in them. The challenges say that a significant number of the people of a community are dissenting from a belief. It may be a belief of which it is said: 'We believe.' It may be a belief of which you or I say: 'I believe.' Do these challenges, if not defeated, require us to retract, or not move to, 'We know,' 'I know,' or both? My answer is that they effectively challenge claims of 'We know' but that they do not undermine all claims of 'I know.'

My specification of 'significant number of people' makes tautological my thesis that these challenges effectively defeat claims of 'We know.' However, it makes more interesting my thesis that they need not defeat claims of 'I know.' By 'significant number of people' I mean: Enough people of the proper epistemic status have challenged a belief in any of the four ways so that, by our communal standards, no one could responsibly teach that belief as what we know.

Can there be a case in which the social challenges defeat a claim that we know while not defeating a first personal singular claim to know? Suppose you are someone who despises a local clergyman and you have expressed your contempt in conversation. Suppose further that a vicious, inaccurate open letter expressing your style of contempt is posted on his church bulletin board. No one has dared to sign this scurrilous note. You are accused. All of the evidence points to you. Your community knows you had a motive. You had been apprehended on previous occasions while doing this kind of cowardly deed. You were seen around the church shortly before and after the letter was discovered. Investigation shows that it was prepared on your typewriter. However, you did not do it.

You protest your innocence. Your responsible neighbors rightly point out that such testimony from an accused is unreliable. Your few friends, who will not accuse you of lying, remind you that even your introspective data on which you base your plea of innocence could mislead you due to a wish that you had not done the deed. Under these conditions no one is warranted in saying: 'We know that you are innocent.' Not even your mother, who still loves you despite all of this, is warranted in consoling you with: 'We know you are innocent.' She is not even warranted in saying: 'I know you are innocent.' Mothers' faith in the goodness of their children is not reliable under these kinds of conditions. *Not all first person singular knowledge claims are protected from the social challenges.* No third person, singular or plural, is warranted in claiming knowledge of your innocence. Nevertheless you did not do it, you know that you did not do it, and can honestly say of yourself: 'I know that I am innocent.' *Some first person singular knowledge claims are immune to the social challenges.*

This example shows that an 'I know p' claim can be protected, and thereby maintained, under social conditions which defeat any 'We know p' or 'It is known that p' claim. I recognize, though, that these social conditions place severe limits on what you can say with your 'I know.' It can be used only for you to assure yourself of your innocence. However, it is of extreme importance for each of us to know ourselves; especially when our community misunderstands us. Let us say that first person singular knowledge claims which can stand against the social challenges give knowledge for self understanding.

This protection of first person singular knowledge claims can be extended. We can extend it to protect claims that I know p apriori. It might be that every claim of the forms 'We know p apriori' and 'It is known apriori that p' are defeated. Nevertheless for individuals to express their understanding of their epistemic status vis-a-vis p, they may assert: 'I know p apriori.' At least the protection can be extended under Kitcher's definition of 'apriori.'

Kitcher's definition of 'apriori knowledge'

To define 'apriori knowledge' Kitcher introduces[9] the phrase 'life sufficient for X for p.' This phrase means that X has enough experiences to understand the concepts of p. Where he uses this phrase we could use 'understand p.'.

Here it is very important for the goals of this essay to note that use of X shifts attention to analysis of third person singular knowledge claims. I suspect that 'X knows p' is "logically closer" to 'we know p' than is 'I know p.' If p is not about the inner life of X, an 'X knows p' can be tantamount to an 'It is known that p.' In many, if not all, contexts 'It is known that p' is synonymous with 'We know p.' I shall alter Kitcher's definition to focus on first person singular claims. I concede to Kitcher that no 'X knows p apriori' holds for mathematical p. With that concession, I also concede that for mathematical p we should not assert either 'It is known apriori that p' or 'We know p apriori.' Examination of his definition shows that I need not make such a concession for the first person singular.

He proposes the following analysis of apriori knowledge as supported by warrants which are "ultra-reliable" and "never lead us astray."

> X *knows apriori* that p iff. X knows p and X's belief that p has been produced by a process which is an apriori warrant for p. W is an *apriori warrant* for X's belief that p if and only if W is a process such that, given any situation in which X understands p, (3a) some process of the same type could produce in X a belief that p,
> [*Producibility* whenever p is understood.]
>
> (3b) if a process of the same type were to produce in X a belief that p, then it would warrant X in believing that p.
> [*Unshakeability* whenever p is understood.]
>
> (3c) if a process of the same type were to produce in X a belief that p, then p.
> [*RELIABILITY* whenever p is understood.]

Full appreciation of such complex analyses requires thinking through a variety of examples and the explaining away of putative counterexamples. We cannot pursue full appreciation. Here. I offer only three comments and a criticism.

First, he leaves to our common sense decisions of whether or not two processes are of the same type. Thus we should class learning a theorem by proving it ourselves and learning it from hearing a distinguished lecturer as learning it by two different processes. But the "symbol manipulation" proofs we go through to learn 'The sum of two odds is even' and 'The product of two odds is odd' are processes of the same type.

I call (3a) the producibility requirement. Clause (3a) rules out almost all, if not all, empirical processes as apriori warrants. Clause (3a) requires that we be able to produce the warranting process in any situation in which we understand p. This rules out direct sensory perception. We can understand what it is like to see a bright red BMW. But merely understanding a situation does not enable us to bring about appropriate sensory stimulation. It also rules out inductive reasoning. We can understand what it is like to have supporting data for a hypothesis. But merely understanding what we want does not enable us to produce it or discover it. Perception and induction need the natural world to be a certain way. Clause (3a) specifies that the understanding is, in principle, sufficient to produce the warranting process regardless of what occurs in nature as long as nature does not destroy that understanding.

Further elaboration of (3a) would bring out that not all of our mathematics could be warranted by producible processes. Theorems requiring very long or very complex proofs are not warranted by processes satisfying (3a). However, I do not need to elaborate on (3a) for my purposes. Nor do I need to argue that all I know of mathematics is warranted apriori by producible processes. It suffices that some processes are producible warrants, eg., what we produce for ourselves when we convince, or reconvince ourselves, that a cube has six surfaces. I want only to argue that some of the mathematics that I know is known apriori by me.

In the third place note that contingent truths can be known apriori. In particular, each us of can know apriori of ourselves 'I exist.' You cannot think of yourself not existing in a situation in which you have the concepts required for being self conscious. As Descartes reminded us: The very act of thinking produces a warranting situation for 'I exist.'So, (3a) is met. (I am uncertain about classifying the Cartesian meditation as an empirical process.) No significant challenge to 'I exist' comes from (3b). In no situation could anyone bring challenges to you which would warrant you being skeptical about your own existence in that situation. Obviously (3c) is satisfied; you do exist in any situation in which you are lead to believe that you exist. Concomitant with 'I exist' Kitcher concedes on p. 30 that we can know apriori such beliefs as: 'I have beliefs' and 'There are thoughts.'

On Kitcher's definition such self-knowledge exhausts our apriori knowledge. Clause (3b) allows nothing else. My criticism of Kitcher is that his definition makes the class of apriori knowledge, in effect, a null class. So, if we accepted his definition as correct we should have to regard all those who talked of apriori mathematical knowledge as talking of nothing, in so far as they spoke correctly. Each of us would be without a term for characterizing our experience of coming to know some mathematical truths with a certain especially satisfying sense that we have insight into why it must be so. Such experiences are crucial for many of us to learn and discover mathematics. Without an appropriate label to mark their special character we lose sight of their cognitive importance. They become confused with the cognitively irrelevant in the streams of consciousness accompanying all thinking. Some venerable words are "worth fighting for." Apriori' is one of them. Kitcher is aware of the occurrence and importance of such experiences in mathematical thinking. But restricting 'apriori' to his way of using it, obscures the importance of such experiences for the growth of mathematical knowledge individuals.

In Chapters 2 through 4, Kitcher analyzes those non- empirical processes which lead so many of us to our mathematical beliefs and which give mathematics a special fascination: Insight into essential features of imaginary and visual structures, Formal and informal proofs, Conceptual analyses. In these chapters Kitcher gives a sensitive and sympathetic phenomenology of individuals' mathematical activity. He does not deny that there are those internal processes, loosely called intuitions, in which we find ourselves engaged when struggling with mathematics. Instead he argues that those processes have only a heuristic function which he labels "local justification."[10]

My criticism of Kitcher can be restated as: A full account of the nature of mathematical knowledge includes an account of the growth of mathematical knowledge in individuals and at the individual level local justification is properly characterized as apriori. How does he argue that they have only a heuristic function?

His ultimate basis for rejecting these familiar, non-empirical and crucial processes in our personal mathematical activity as apriori warrants is the possibility of experiencing those social challenges arising against any mathematical p. Consider what happens, to those of us who have internalized the epistemic standards current in our Western culture, if we experienced one of those social challenges to a mathematical belief. Consider one of the most elementary: 2 + 3 = 5. Those social challenges suffice to show that the community does not believe p, eg., 2 + 3= 5. Hence, we cannot teach, in such a situation, 2 + 3 =5 as what we know. Similarly we cannot legitimately assert: 'It is known that 2 + 3 = 5, or of some third party X: 'X knows that 2 + 3 = 5.' We have, then, a possible experiential situation which shakes conviction that 2 + 3 = 5 is in our body of mathematical knowledge. So, whatever that process W by which we accepted 2 + 3 = 5 might have been, W fails condition (3b) *if it must produce unshakeable conviction that* 2 + 3 = 5 *belongs to communal mathematical knowledge*.

However, what if the process need only give you an unshakeable conviction you have the correct answer or proposition even if you cannot responsibly teach it as the correct result? Can we alter his definition so that it requires only the unshakeable conviction you need to realize that you have reached a correct result in your mathematical thinking and are ready to use it to reach other results?

Why does Kitcher define 'apriori' so that it is, in effect, unrealizable. He required his definition 'apriori' to make precise 'knowledge independent of experience.' Kitcher challenges those who think there is a more realizable notion of apriori. We can read the following.

> "I contend that the analysis of a priori knowledge given in Chapter 1 provides the only clear account of the epistemological notion of apriority which is currently available. Hence if someone wishes to protest that my analysis stacks the deck against the apriorist, it is incumbent upon him to provide an alternative. Given the arguments for psychologist epistemology, rehearsed in Chapter 1, it seems that any such account will have to take the form of specifying conditions on a privileged class of processes which could serve as a priori warrants for belief. If these conditions do not include the constraint that the processes in question be able to sustain knowledge independently of experience, then I think the distinctive idea of epistemological apriority will have been abandoned. ,..." fn.1 p. 89

We can meet his challenge if we restrict the purposes for which we gain the mathematical knowledge we want to say we know apriori.

Part III: *Definition of apriori for first person singular*

Let me offer a definition of 'apriori knowledge' which permits you to describe crucial stages in your mathematical experience in terms tantamount to: 'I know p apriori.' Despite my repeated emphasis on paying attention to first person singular knowledge claims, I use second person singular in my definition! Why? In appraising the definition you should consider whether or not it applies to you. When you so apply it, you will ask yourself: Can I say I know under such and such conditions?

> You *know apriori* that p iff. you know p and your belief that p has been produced by a process which is an apriori warrant for p. W is an *apriori warrant* for your belief that p if and only if W is a process such that,
> a] W produces in you a sense of conviction that you cannot conceive of p being false,
> b] W is critically appraised by you,

and given any situation in which you understand p

> c] some process of the same type could produce in you a belief that p with the same conviction that you cannot think of p being false regardless of what others may say about the truth of p.
> d] if a process of the same type were to produce in you a belief that p, then p.

What are the processes W which satisfy [a]? Tamara Horowitz's "A Priori Truth,"[11] suggested 'ratiocination' as a term for producing those processes in ourselves. I share her view that the processes mathematicians use to certify for themselves that they are correct provide paradigms for ratiocinative processes and apriori warrants. Ratiocination comprises those processes, named earlier, which Kitcher describes so well but, in his discussion of local justification, relegates to individual heuristics. For your own examples prove a geometry theorem, write down and factor a polynominal, prove a trigonometric identity. Such examples will remind you, if you needed a reminder, of the importance of these processes for development of your mathematical knowledge. Until you get these ratiocinative processes to work you say: "I just don't see it; I can't figure it out!"

Condition [b] is vague but crucial. Ratiocination is not merely a process of attaining conviction. It is a process of seeking correct conviction. Recall how it is when you work through a proof. You analyze your procedure to ensure that you have not made errors such as overlooking cases or adding assumptions. A characterization of how we correct ourselves when we are merely working out a problem for ourselves goes way beyond the scope of this paper. I am finally beginning to appreciate Wittgenstein's insight in his *Remarks on the Foundations of Mathematics*.

> I have not yet made the role of miscalculating clear. The role of the proposition: "I must have miscalculated". It is really the key to an understanding of the 'foundations' of mathematics. II-90 RFM [12]

At least for understanding the growth of mathematical knowledge within individuals we need to describe and appraise our standards for self-criticism. Of course, standards of self- criticism are learned. Nonetheless, these standards allow us to resist pressures from the very society in which we learned the standards. This leads to comment on [c] above.

If you work carefully and critically through a proof of some rather elementary mathematical claim, you are rightly convinced that it cannot be otherwise. You can imagine those social challenges arising without shaking your personal conviction. You would simply work through the proof again and have the same sense of it having to be so regardless of what the other people said. You would know how it is with your own mathematical knowledge. (Now there is a place for talking of your own mathematics because it does not cohere with communal knowledge.) This sense of being correct despite what others say is valuable for self understanding just as it is crucial for innocent people to realize their innocence when all accuse them. In the mathematics case the personal conviction is necessary if you are to struggle to convince others of your views. Of course, you must admit that your insight could mislead you. Even if you cannot "see" how you could have "miscalculated" you must admit that you could be cognitively defective in some way. So, without understanding how you could be mistaken, you have to admit that your intuitions are fallible.

With this concession of the fallibility of my intuitions - what I know apriori- I close these relections. Let me, though, end with a speculation about the singular and the communal. Perhaps we can never develop a single unit (machine) to produce all of mathematics. Perhaps to get all of mathematics we need to develop a community of machines. Our inability to produce a community renders us unable to replicate our mathematics as the output of a single unit. We are a community and cannot replicate ourselves.

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NOTES

1. Oxford U. Press, New York 1983. All references to Kitcher will be to his *The Nature of Mathematical Knowledge*.
2. *Journal of Philosophy* 81(1984) pp. 449-456, p. 452
3. I have been exploring the themes of Kitcher's book for some time. In Fall of 1984 I was introduced to it in a Ohio State University reading group lead by Stewart Shapiro. In August 1987 I presented "The Mathematical Apriori after Kitcher's Critique" to the 8-th International conference on Logic, Methodology and Philosophy of Science, Moscow U.S.S.R. See pp. 36-39 Vol. 1 of the conference *Abstracts*. In the Moscow paper, I despaired of finding a place for apriori knowledge. I made only a plea for accepting apriori thinking in mathematics. In Fall 1987 I used Kitcher's book in an Ohio State University philosophy of mathematics course. Finally I realized the value of distinguishing between knowledge at the communal level and singular knowledge. I developed my hypothesis that Kitcher's dismissal of apirori knowledge applies only to communal knowledge. I presented this hypothesis in "Fallible Intuition: An Alternative to Kitcher's Apriori" to the Ohio Philosophical Association, Xenia, Ohio, Ap. 9, 1988. See the 1988 OPA Proceedings pp. 12-21. At the OPA meeting, Professor James Taylor of Bowling Green State University presented helpful criticisms. The present essay is a drastic revision of the OPA presentation. Here I focus primarily on the distinction between singular and communal knowledge rather than on the definition of 'apriori' for singular knowledge.

   Readers should be aware that my extensive philosophical involvement with Kitcher's book may make me an unreliable interpreter of his views. My interpretation is influenced by my judgments of "what he should have said," "what he really meant," etc.,.
4. I accept *Philosophical Investigations* 127: "The work of the philosopher consists in assembling reminders for a particular purpose" as characterizing a fundamental form of philosophical argument.
5. My remarks on supposition are based primarily on the critical analysis of an early 13th-century English logician's,

William Of Shyreswood's, *Introductiones in Logicam,* by William and Martha Kneale on pp. 246ff of their *Development of logic*, Oxford 1962.

6. Pp. 337-363 *Evolutionary Epistemology*, Werner Callebaut & Rik Pinxten (eds.), Reidel, Dordrecht, 1987
   Professor Van Bendegem makes the distinction. He is not responsible for any ontological interpretation of the distinction which I may make.
7. *Die Philosophie des strikten Finitismus*, Ernst Welti, Peter Lang, Bern 1986.
8. Kitcher, pp. 22ff.
9. Kitcher pp. 22ff. I do not quote his analysis. Nonetheless I keep his numbering (3a) etc.. I think that for some students of Kitcher's book these numbers have become proper names of the conditions. I add the bracketed comment beneath each condition.
10. Kitcher pp. 94ff.
11. Journal of Philosophy 82 (1985) pp. 225-239.
12. Eds. G.H. von Wright, R. Rhees, G.E.M. Amscombe, trans. G.E.M. Anscombe, Macmillan, New York, 1956.