BETWEEN PHILOSOPHY AND MATHEMATICS:
THEIR PARALLEL ON A "PARALLAX"

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1. *Retrospect and prospect*

There was a day, barely half a century ago, when T. Takagi[1] of
the Class Field Theory fame called one[2] of his star-pupils a
"lazybones"[3] – all because the latter had decided to specialize in
the foundations of mathematics, and this, even in the post-
Gödelian days. The "working" mathematicians had, and still have,
hardly anything to do with the extremely specialized area of
research today: "Foundations of Mathematics" or more specific-
ally "Mathematical Logic" (attested of late by none other than an
old master, Saunders MacLane[4]).

This sort of feelings in alienation is nothing new. Kant's first
Critique, for one, began with the lament of Hecuba (Aviii-ix),
quoting the original Latin form *Metamorphoses* of Ovidius.[5] To
wit: the "spin-off" (if spoken in the current TV lingo) of the
once almighty Philosophia qua Regina Scientia has grown in such
a large number that she now feels "forlorn and forsaken" and
even entirely forgotten! In the days of Socrates, for instance a
"mathematician"[6] was simply inconceivable without being a "phi-
losopher" as well. This tradition in the West was quite explicit
up to the days of Descartes and Leibniz (and even the old
Newton tried to write a learned commentary on one of the
Evangelia). All mathematicians in those good old days were no
doubt some sort of philosophers to a certain degree (that is, not
quite professionally), and vice versa, perhaps to a lesser de-
gree.

It was but natural, in fact, that no alternative was possible
within so manifest an academic tradition of artes liberales
(meaning the nearly two millennia old[7] Trivium plus Quadri-
vium). As late as the last century, indeed, some brilliant stu-
dents had to leave Cambridge or Oxford without a degree be-
cause of their weakness in mathematics.[8] In the Eighteenth
Century, then, Kant for one had to dabble with mathematics and
later, as Privatdozent, had to teach a few times a course titled

"Mathematik" for making a living. And all this, with very little "workable" knowledge of mathematics[9] - not too unlike Christian von Wolf,[10] but certainly very different from Leibniz. The level of his mathematical knowledge was then only slightly above the range of a Rechenmeister[11] a century or so earlier. (Note here the importance of "rechnen" in Wittgenstein's philosophy of mathematics.) He had acquired, however, at least the popularized sort of Newtonian mechanics;[12] thence began his philosophy of mathematics and physics, the first *Critique*.

The tradition of a philosophy of mathematics "in medias res" sorts (to enter "into the midst of things" without, in the present case, mastering the subject to do philosophy about it) was firmly established then and there, when and where the first *Critique* had turned into an immortal classic for all subsequent studies in the philosophy of mathematics. Moreover, what Hecuba once feared most had grown so routine that differentiations (sub-sub- ... -sub-divisions) of the general human knowledge cannot but turn into an Academic Assembly Line - to such an extent that "scholia" (commentaries on commentaries ... on a particular philosopher, dead or alive for instance - in themselves may be mistaken for "philosophia" itself today.

Not, however, that there is anything amiss or wrong about scholia in themselves. If scholiasts, erudite or profound, did not exist, they certainly must be invented. If scholia on Kant's (or Wittgenstein's, for that matter) philosophy of mathematics must be considered "the" philosophy of mathematics, however, they may have to be taken with a very large grain of salt. For, needless to add, the "scholia on Kant" have been well-known, in or for themselves, to be notoriously inadequate if not incorrect outright. If otherwise, how can one justify the awesome height of an enormous mountain (ever growing higher, monthly, yearly!) of Kant-literature - ditto, of Wittgenstein-literature!?

To name but one (out of hundreds, thousands), the most recent paper on Wittgenstein's philosophy of mathematics was almost exclusively concerned with his philosophical "remarks" on "counting" ("rechnen").[13] That is, as if "the" philosophy of mathematics could be exhausted by philosophical remarks (whatever they may be) on "counting"!? That is, again, as if the growth of mathematics had completely stopped in the long-gone days of Rechenmeister!?

The latter approach may make perfect sense, of course, if Hilbert for one had actually succeeded in the "arithmetization" or "algebraization" of [Euclidean] geometry *in toto* and *in concreto*, notably in his epochal work in 1899.[14] Nothing of the sort (or even remotely similar to it) has ever taken place, however,

contrary to the misguided or perhaps indifferent belief of some. To wit, the time-honored dichotomy between "rechnen" (for arithmetic-algebra) and "zeichnen" (for geometry) has never been dissolved. A recent book on Wittgenstein (an admirably erudite work in or for itself)[15] snickered or all but snarled at the "naive definition", adequate for the dictionary and for an initial understanding" of mathematics routinely given by two working mathematicians turned philosophers, [16] namely: "the science of quantity [for arithmetic-algebra, to "rechnen"] and space [for geometry, to "zeichnen"]."[17] As if, to repeat, "the" philosophy of mathematics could be exhausted by the exclusive studies on counting or constructing the formal [arithmetic or geometric] systems as axiomatic structures or studying the nature thereof in the name of foundations of mathematics!?

What or how about the nature of the quintessential difference between "rechnen" and "zeichnen" itself? Why the dichotomy in the first place? In which epistemological sense? And, if only to be on a par with the philosophy of physics *today*, how about all these problems in terms of mathematics *today*, for instance in terms of algebraic topology and topological algebra? The philosophy of physics had long since stopped to be monolithically preoccupied with the problem of "measuring" and some other related problems *alone*. If so, why so much [almost absolutely exclusive?] preoccupation with "rechnen" (in many-splendored set-theoretic language or otherwise) in *the* philosophy of mathematics *today*?

2. *Mathematics and metamathematics*

The so-called "set-theoretic incest"[18] or more generally "self-reference" is so well-known today (how else in the presence of paradoxes in hundreds!?) that it is unlikely for any logically-minded student commit that particular kind of cardinal sin. After all, even over two centuries ago, Kant saw to it that, while liberally and at times almost carelessly employing his key-term "transcendental" throughout the first *Critique*, the latter was only for his philosophy or a *theory*, but never (never ever once!) for his *method* to construct the theory. If so, why such an un-theoretical indifference toward the mistaken identity between "mathematics" and "meta-mathematics" relative to "the philosophy of mathematics" today?

In the aforementioned, indubitably very erudite scholia on Wittgenstein [19] for one makes itself fairly conspicuous by the totally erased or blurred lines between "his" (Wittgenstein's)

and "the" (entirety of) philosophy of mathematics, between "foundations" and "philosophy" thereof (or of mathematics itself as a whole), and the most blurring of all, between "metamathematics" (or "meta–meta–mathematics" for that matter) in particular and "mathematics" in general.

There was once an indiscriminating mixing–up of the whole and some parts thereof in the name of a "New Math" two decades or so ago as if not "pars in toto est" but rather "pars pro toto" must be the only proper case. Namely, doing mathematics (in the American classroom) in the spirit of pars pro toto: study logic to understand mathematics since the basic principles of mathematics are reducible to logic (with the abundant aid of something extra–logical)! And this, too, in terms of quasi–"set–theoretic" language.[20] This, of course, is a twice–told tale, and long, long ago at that. In "the" philosophy of mathematics, however, no similar practice seems to attract any attention from the side of its practitioners in general.

This type of modus vivendi of indifference it must be, then, that loves to name–drop such illustrations names of working mathematicians as G.H. Hardy[21] of the "pure mathematics" fame while omitting his no–nonsense view of mathematics: "All physicists, and a good many quite respectable mathematicians [including Hardy himself no doubt!], are contemptuous about proof. I have heard Professor Eddington...maintain that proof, as pure mathematicians [do.] understand it, is really *quite uninteresting and unimportant*, and that no one who is really certain that he has found something good should waste his time looking for a proof ..."[22] And: "In the first place, even if we do not understand exactly what proof is, we can, in ordinary analysis at any rate, recognize a proof when we see one".[23] (These remarks are found in his book on *Ramanujan*, 1940, no doubt written mainly for working mathematicians; the latter, in fact, are still very much interested in some of Ramanujan's conjectures today).

Beweistheorie for metamathematics is certainly concerned with a theory of proofs or even proofs of that theory itself, which for one may amply exemplify the quintessential difference between mathematics and its foundational studies, metamathematics. The *"once for all"* – Alles oder Nichts – is the soul of metamathematics while that of mathematics proper is more or less *"one at a time"* (unum post aliud), always ready to by–pass or side–step some insoluble spots for the time being.

No working mathematician can make head or tail of the Axiom of Choice, but if need be, s/he may freely employ it for a certain result for the time being. What must be recalled here is the Principle of Incomplete Knowledge, best (or at least most quota-

bly) described by Otto Neurath: "We are like the sailors who must rebuild their ship in the open sea without ever being able to take it apart in a dock, putting it together nevertheless anew out of its best parts."[25]

Euclid, for one again, could not make head or tail of the notion of infinitum (to apeiron). Hence the total absence of the latter throughout the entire theirteen books of his *Elements*. Hence, too, his ambivalent location of the Fifth Axiom (of Parallels) within his formal structure of geometry. It was simply inconceivable for Euclid, to be sure, that he would throw up his arms because of that single topic of ambivalence, or that he would single-mindedly concentrate on a certain "aporia" for its solution "once and for all" and for ever, without grappling with the main body of geometric (and arithmetic) knowledge. Without any hesitation he would rather bypass or sidestep the aporia even though, because of such a bypassing or sidestepping, he would have to satisfy himself with an "incomplete" systtem. Similar attitude abounds throughout the entire history of mathematics;[26] nay, it has been the very routine modus vivendi or operandi of all working mathematicians since Euclid or even earlier.[27]

As a matter of fact, the "uncertainty" (that Morris Kline,[28] or rather his impatient critiques, recently made a federal case of) has always been at the core of mathematics if seen locally. That is, the global or rather Quixotic concern over the absolute (!) certainty of mathematics is of relatively recent origin, notably since Hilbert[29] and his Metamathematik. The greatest emphasis at this point is on the common misunderstanding that the generic term "mathematics" was in wide use even in antiquity while, in truth, the generic terms even in the mediaeval centuries were "trivium [grammar, dialectic and rhetoric] and quadrivium [*individually*, geometry, arithmetic, spheric, and music] of [seven] artes liberales" rather than the general "mathematica" in Latin. There never was a day of "[foundational] crisis" for working mathematicians who, having hardly any time for the Hilbertian doom-boom, went on "doing business as usual" hourly, daily, weekly.

The anecdote of Einstein about Hilbert and Brouwer, "What is all this frog-and-mouse battle between the mathematicians about?", was widely popularized and immortalized by 1939.[30] This feeling of lofty indifference was shared more or less with just about all working mathematicians, contrary to the voluminous (and sometimes shrill) writings by the "experts in foundations" in particular for over eight decades (most loudly in the last five). This fact of common and routine practice by working

mathematicians may be perhaps more clearly and easily under-
stood if considered together with such well-known anecdotes as
those about Dirac,[31] Heaviside,[32] etc. (not necessarily, but gene-
rally more in the schools of applied mathematicians, even though
enough can be found in that of pure mathematicians, too).

If nothing else, at least one modus operandi among working
mathematicians, quite explicitly distinguishable form that of
meta-mathematicians or logicians, must be repeatedly pointed out
and emphasized: since Euclid or earlier, the former was always
at work in the spirit of *one-at-a-time* for a specific problem at a
particular time. The latter, on the other hand, had to be con-
cerned with a far more philosophical (than properly mathemati-
cal) aspect, *once-for-all* even in principle, of the foundation
"sub specie aeternitatis" in essence. With this much explicit and
distinct difference in their approaches to the nature of mathe-
matics, then, mathematicians and metamathematicians or logicians
could not possibly be concerned with the same aspects or types
of philosophical problems. And, in the light of history, they did
not and still do not.

This kind of distinction may be viewed, perhaps (if seen
roughly), in terms of "apples and oranges" or in the language of
William James, of "tender and tough-minded" approaches in
philosophy. Figuratively speaking (not necessarily in the actual
terms of individual personality or characteristics) in terms of
the last distinction, mathematicians are tough-minded and meta-
mathematicians or logicians tender-minded; the former *in gene-
ral*, "empiristic (going by 'facts'), fatalistic, ... pluralistic, scep-
tical"; the latter *in general*, "rationalistic (going by 'princi-
ples'),... free-willistic, monistic, dogmatic"[33] in some particular
sense that need a certain interpretation as below.

The first characterization is hardly in need of explanation in
that the "facts" for mathematicians are indubitably the actual
body of mathematical knowledge, consisting of established theo-
ries and theorems therein. One at a time, even if only for the
time being, to grapple with some specific problems presented at
some particular times (under certain "conditions" or "limita-
tions"). This fact was perhaps most strongly emphasized by none
other than Georg Cantor himself, and since the following lines
have always been somewhat (farcially?) shortened and, con-
sciously or unconsciously, misrepresented and misinterpreted,
they are now shown in the original German in its entirety:

In the eighth section of "Grundlagen einer allgemeinen Man-
nifgaltigkeitslehre" (Leipzig, 1883), Cantor wrote: "Die Mathe-
matik ist in ihrer Entwicklung völlig frei und nur an die selb-
stredende Rücksicht gebunden, dass ihre Begriffe sowohl in sich

widerspruchlos sind, als auch in festen durch Definitionen ge-
ordneten Beziehungen zu den vorher gebildeten, bereits vor-
handenen und bewährten Begriffen stehen."[34] After this much
emphasis on the thorough familiarity with or mastery of the
entire body of mathematical knowledge in the past and at pre-
sent, *then (and only then)*, one may start to stand on his own,
"freely" creating (and building the body further bigger and
sounder). So, after that much clarification in concrete, he added
(about twenty lines later): "...denn das Wesen der Mathematik
liegt gerade in ihrer Freiheit."[35]

The "free-willistic" and "dogmatic" tender-minded approach,
however, was such that only the last line has been lovingly
quoted, indeed grossly out of context, by many a set-theorist or
expert in the foundations of mathematics, to say nothing of some
"philosophers of mathematics" following in the footsteps of the
former. The "tought-minded" approach, on the other hand, has
always been more "fatalistic" than less, resigned to the "Fate" of
Incomplete Knowledge, practically or perhaps pragmatically con-
cerned always with "empiristic" (going by [mathematical] 'facts')"
treatment of specific problems at particular times. If so, (and
even if not, at least in the spirit of "the taste of the pudding is
in the eating"), one must consider the difference, or at least the
possibility thereof, between the philosophy of mathematics and
the philosophy of metamathematics, preferably with respect to
some specific problems in concrete, as below.

3. *"Rechnen" versus "zeichnen"*

Leaving many (occasionally fascinating) problems directly re-
lated to the foundations of mathematics or metamathematics (if
only "for the time being"), one may face the most familiar
paradox of all, Zeno's from the standpoint of mathematics proper
via a "Verstehen"-approach, namely: Put oneself into the sandals
of Zeno himself. The latter had absolutely no inking of any
set-theoretic language or any refined theories on "continuum"
(known to us today), needless to add. All he knew or cared were
arithmetic-algebra (with fractions) and geometry, as far as his
own paradox was concerned, and all these, absolutely only in
terms of the very natural "natural numbers" (literally in
Kronecker's view of God's creation).

The "motions" of Achilles and tortoise in actuality were obvi-
ous to observe (or perceive) and could easily be "drawn" (zeich-
nen) in figures, geometric or otherwise. Their race became a
paradox or at least an aporia, however, as soon as Zeno had

begun to consider the same problem in terms of one-to-one in the (conceptual) language of arithmetic–algebra. Or, if seen in comparison with a similar problem, the drawing of a square with unit length as its side was the easiest for anybody, even for a moron, to see (or perceive). Ditto, the drawing of a diagonal of the square. To compute (or conceive) it in arithmetico–algebraic terms, however, was something else. For, even in the primitive proof (within the domain of only natural numbers, without any infinite number of counter-examples exemplified by a division by zero), it instantly brought forth the insoluble problem of incommensurability, an aporia.

None of such an aporia as above had ever fazed any "working" mathematicians in general (or rather, in reality, individually and specifically geometers and arithmetico–algebraists); they went on, business as usual, solving one specific problem after another within their reach. Since pre-Euclidean days onward, then, the main or perhaps only motto for working mathematicians has always been: Divide and conquer (with a certain hope of "fatalistic sceptics" expressed best, some two millennia after Euclid, by d'Alembert: "Allez en avant, et la foi vous viendra")!

Not the monolithic whole, indivisible and ultimately incomprehensible totality, of "mathematica" (let alone, more philosophically, "matheses universalis"), but the concrete and separate geometry and arithmetic–algebra were for the practical division of labor from the very beginning of mathematics as an exact science in the Euclidean format (of an axiomatic structure firmly founded on a physical, rather than any abstract, foundation). It was not unheard-of therefore that such a division as above was even incorporated into the study of "national [or ethnic] characters" such as French versus German (and the like, not always tongue-in-cheek either). Some old-timers among us can still vividly recall the "scandal" caused by just such a study by Ludwig Bieberbach after the rise of Hitler in Germany.[36]

The last circumstances made any topic on "mathematics and national character" or even "mathematics and personality" and the like a sort of "verboten" since the end of WWII. Since these topics had existed much earlier than the present century, and indeed absolutely without any Nazi-like ideologies, they deserve to be resurrected if only to prove beyond a shadow of doubt that there is something more than an ideology and a "despicable" racial or ethnic prejudice. For one, until they were monstrously distorted and tainted by the Nazi- or quasi- Nazi ideologues, they had always reassured some (many if not most mathematicians?) to consider themselves "algebraists" or "geometers" (or "topologists" today) or "analysts" rather than

broadly "mathematicians" in the most generic sense.

Without getting involved in any ugly (and occasionally hysterical) controversy, but simply to resurrect the time–honored practice (since as early as Euclid or even earlier) in the history of mathematics, however, such a "divide and conquer" as above entailing the "personal character or talent"[37] or "certain (national or regional) schools and traditions"[38] may attain a new respectability – if only to correct, in no uncertain terms, the current emphasis on the "comeback" of geometry[39]. And this, when geometry was never ever away, let alone "dead"!?[40]

The division (of labor in the practice of working mathematicians at least) between certain branches of mathematics was made absolutely clear as late as 1940 when Hermann Weyl did not mince his words: "In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain." This may not be as profound as the division or opposition in the sense of "coincidentia oppositorum" of Nicolaus Cusanus,[41] but what is self-explanatory here is the "alive and kicking" state of the fundamental distinction between "rechnen" and "zeichnen" or between the conceptual and the perceptual or between the algebraic and the topological (or geometric).

4. "Parallaxes" rather than "paradoxes"

If seen in the light of the above, much or most of the "solution" of Zeno's Paradox must be considered either totally missing the point or inadequare at best, in that such solutions have always been of the type to force the language of apples down the throat of oranges. And this, never even beginning to understand the need of more studies on the "parallax" between the arithmetico-algebraic and the geometric, or between the conceptual and the perceptual.

The latter type of bifurcation itself may make sense only to the "tough-minded" approach, while making very little sense to the "tender-minded" one. (How radically different these diametrically opposed approaches can be may be observed in a single paper on Descartes, for example in a recent philosophical journal.[42] A tender-minded ontological approach to Descartes can prove once and for all that apples and oranges can never if ever communicate to any intelligible degree). If so, the following arguments must be directed to the tough-minded only.

Since a growing number of educational psychologists[43] have of late broaden their studies on "talent" or "intellect" (rather than

"intelligence" of IQ sorts) so as to distinguish the "logico-mathematical" talent from the "spatial" among several others, these "experimental" data may be incorporated into the problem at issue here. If nothing else, the bifurcation of the conceptual and the perceptual, or the arithmetico-algebraic and the geometric, may be "established" here as a reasonable hypothesis originating from the ancient ages of Egypt, Babilonia, etc., and more recently and systematically, from the context of Kant's first *Critique* in particular if not *De docta ignorantia* of Nicolaus Cusanus.

Within this context, then, the "parallax" between "intuition" and "understanding" of the human(!) faculty is fait accompli. In the divine (!) faculty, if only per definitionem via per analogiam, these two can perfectly, completely, instantly, eternally "coincide" in absolute terms (forcing Kant to coin such a term as "Intuitive Intellect" or "Intellectual Intuition" for God the Almighty, hesitantly employed here and there between 1770–81, if only to exemplify in concrete the "limits" of human faculty in analogical comparison with the Godly[44]). The next problem for Kant was: How can the two interact between themselves (of the trio, adding the highest tribunal, the "reason" that was a part of "intellectus" in 1770)? Or how close can they come to each other while interacting, complementing?

First, just as he introduced the "power of judgement" (missing "power of" in all English translations) as the sole intermediary between understanding and reason, he bridged the chasm between understanding and intuition with the "power of imagination (do.). Moreover (perhaps only an "architectonic" improvement?), he introduced two key terms "intuition in general" (new) and "consciousness in general" (equivalent to the old "transcendental apperception" in understanding) in the second edition of the first *Critique*, hoping (?) that they could somehow come closer together in terms of "in general" (in conjunction with "experience in general, things in general [instead of its equivalent 'things in themselves'], and many other key terms now stipulated with "in general"[45]).

After so much ado about bridging intuition and understanding in this manner and that, however, the first *Critique* merely ended up with all the more clearly visible presence of "parallax"[46] between the perceptual of intuition and the conceptual of understanding. If so, so be it! It only justified a fortiori the original and primary purpose of investigating the "source and boundary" of the Reason or the Human Faculty as an organic whole, the very raison d'être of Kant's Critical Philosophy. Never to be shaken to the slightest degree after all the bridging

efforts, then, was Kant's conviction, declared (and oft-quoted) in the first edition of the first *Critique*: [The percepts of] intuition without [the concept of] understanding is blind, and the latter without the former, void.

Long live the parallax, intrinsic in the human faculty or in the manner of pursuing knowledge by human beings, characterized by "hominis est errare" (Cicero, *Orationes Philippicae*, XII, 2,5) or "errare humanum est" (Hieronymus, *Epistolae*, 57, 11)! After all (the best and greatest effort by the best and brightest of all human beings for the last several millennia, founded on the brute experience since the first homo erectus or even earlier), the human knowledge as a whole is barely comparable with a jigsaw puzzle consisting of billions, trillions of pieces. Amazingly large parts of "the puzzle" have already begun to show us a fairly plausible-looking picture of the whole, here and there with some exquisitely, exactly "dovetailing" pieces to show us some "elegant" or profound solutions, it is true, but still only here and there at best – always better locally, and never globally as a whole. This, then, is the quintessential nature of the human knowledge (as the tough-minded students understand it), inevitably always incomplete due to the parallax intrinsic in the human faculty itself.

5. *Algebraic topology versus topological algebra*

A noted Belgian algebraist[47] wrote once to the effect that the situation in 1940 described by Weyl had dissolved itself by late in the 1970s since, as he saw it, algebraic topology had more or less completely subsumed toplogical algebra under itself or vice versa. But, then, how much or little is "more or less"?

How exactly, and why, precisely, are they interacting in detail? And if in detail, where, most clearly, can one pin-point the critical [interdependent] areas that refuse the ultimate dissolution of "parallaxes" between the algebraic and the [ultimately] geometric or the topological? Such questions as these can be answered only by those engaged daily in the studies of both at their dizzily explosive research front (where, it was said[48] even in the mid-'60s, one could not return to the mainstream after some three months of absence); that is, if the very best "up-to-date" knowledge alone is required. For the philosophically-minded, however, no such very "up-to-date" state of the art is necessary; a general survey of these two swiftly growing branches of mathematics up to the '70s will do, if only to justify the presence *today* of the good old "parallax" between

arithmetico-algebra and geometry in a brand-new cloth.

A philosophically-minded algebraic topologist or topological algebraist can carry out such a survey without too much difficulty, but to the best knowledge of the present writer,[49] it has so far failed to appear. And there is certainly no place here, severely limited in space, to try even the most crude first attempt.[50] Suffice it here to add only the following, especially for those interested in the question, officially raised over two decades ago:[51] "What is, and ought to be the philosophy of mathematics [today – somehow, somewhat comparable to the level or scope of the philosophy of physics, biology, etc., *today*]?"

In this day and age of computers with their magnificient "graphics" and "fractals" above all, it has become just another matter of fait accompli to see the "comeback" of geometry. But if the same term is now[52] conspicuously employed by a "working" mathematician turned philosopher (a rare breed, indeed – thank God for that!) to explain the "geometric" (or rather topological) aspect of Thom's catastrophes, one cannot help feeling "short-changed" or worse. Worse, in that one feels confused as well when the "qualitative" (versus "quantitative" and "topological") aspects of Catastrophe Theory are somehow or other re-emphasized here as if they had never been self-explanatory!

If the Hegelian-Marxistic "aufheben" (sublation) from quantities to qualities is to be disregarded, the traditional dichotonomy of "qualitas versus quantitas [qua numerus]" has always been of a mutually exclusive kind like any pair of disjoint sets. Since the (conceptual) arithmetico-algebraic has always been considered quantitative, no (exactly dated) predictions of any forthcoming catastrophes can possibly be computed by Thom's theory. This must be self-evident to anybody who has some inkling of what has been going on at the research front since Euler (let alone Poincaré et alii, et aliae, in the last century).

Only the over-zealous type of social or human scientists who would jump at any bandwagon that might give their studies the precious appearance of "[exact] science" must have mistaken the theory of catastrophe for a quantitative science. And the aforementioned mathematician turned philosopher had to spend the entire book of 146 pages to tell us that "it simply ain't so". Meanwhile this author, like several of (the very few) like-minded popularizers in the past decade, did not forget to add a chapter titled "The Comeback of Geometry" as if the "qualitative" (or topological) aspect of mathematics, the time-honored "geometric spirit" since and before Descartes, had ever left the center stage of mathematics! Since his voice was not quite one "in the wilderness" in the world of mathematics, however, a little more

attention may be called here for, as below.

6. *The eternally geometric*

The present century has so far had an "axiomatic" crapluance of
sorts – a monomaniac hung up on the most stringent rigorism,
more likely a hangover from the intoxicating thrills of "axiomatic
thinking" at the turn of the century, Hilbertian or otherwise.
For, without this kind of perhaps all too far-fetched generaliza-
tions, one cannot possibly understand why, typically in the
manner of an Academic Assembly Line, many if not most of
teachers in higher mathematics have almost believed in the
"death"[53] of geometry on the level of undergraduate mathematics.
And this, while an undergraduate course in elementary topology
(sometimes explicitly, algebraic topology) has been offered
throughout the nation.
  These facts of mathematical life in education may have been all
for the best if its humble "empirical"(!) origin and heritage had
not been so completely forgotten or scoffed at. After all, the Joy
of Logic (or Formal Reasoning) in Abstraction had always been
exquisitely elegant, infectiously seductive, hence deliciously
appetizing. By the time of WW I, then, even Felix Klein (once the
supreme geometer of Erlanger Programme) was heard to say,
perhaps resignedly,[54] "The three great[55] capital A's in mathemat-
ics: Arithmetic, Algebra, and Analysis are now reduced to a small
letter: a, meaning 'abstract'." Was this kind of remarks enough
to mislead all(!) philosophers and some(?) historians of mathe-
matics to neglect the honorable origin and the glorious heritage
of mathematics proper in entirety?
  While the next century is only a few years away, the French
way of enumeration (in France proper, not necessarily in other
French-speaking nations) is all the more clearly if not so
strongly hinting the smell of their toes as it was a millennium
ago. If in doubt, start counting in French from seventy to
ninety-nine in that clear-cut vigesimal system. If a somewhat
more abstract (hence more conceptual) "rechnen" can still smell
the flesh, is it any wonder that the more down-to-eath ("geo-")
"zeichnen" in geometry the perceptual cannot but smell The
Good Earth as strongly as ever?
  Or returning to the dichotomy of qualitas versus quantitas,
one can certainly imagine the ancient scene of "measuring the
earth" with some primitive know-hows of a (proofless, still more
or less hunch sort of) Pythagorean triangle and the like. If
"the" Pythagorean Theorem was consciously employed at all, the

people in antiquity still had absolutely no way to compute (in quantity or number) any general Pythagorean-trios.[56] That is, the Theorem had to remain quintessentially "qualitative" (or topological in the language of our times) or "geometric" to the letter.

At this juncture, then, one may as well review the earlier case (in the third section above) for drawing a diagonal of the square with four sides of unit length to recall how totally free from any aporia it was as long as it remained a purely and properly geometric problem. Whas it perhaps (from this word on, an entirely irresponsible speculation!) this kind of geometric intuition in the simplest that gave Fermat the delusion to in the validity of his proof for the Theorem named after him?

As if it were a "Fermat's revenge" (or whatever), his Great Theorem is still being attacked, at least partly, by a geometric approach today, as was conspicuously exemplified by the recent near-earth-shaking event of a (faulty) proof via Geometry of Numbers (or Arithmetic Geometry)[57] by Yoichi Miyaoka on 26.2.1988.[58] What was proved, incidentally but simultaneously, was the glaring fact that Geometry had never been away or half-dead to be in need of "Comeback" in the first place.[59]

The bed of mathematics may be occasionally procrustean; after all, mathematics cannot possibly handle everything, everywhere, all at once. It must be an illusion, however, that the femme fatale of Geometry the Perceptual will ever be out of sight even from the procrustean bed in any time soon. If so (even if not so, at least if "tough-minded" enough to take the bifurcation of percepts versus concepts in one's stride), the philosophy of mathematics "ought to" take up the problem of their parallax with all the seriousness the latter amply deserve, to say the least, for an initial exploration.

<center>∗   ∗   ∗</center>

Last but not least (for some tangible, very well-established world-wide, facts against such amateurish emphases as "the Comeback of Geometry" if only for education in undergraduate mathematics). The International Congress of Mathematicians in 1990 is to held at the University of Kyoto, Japan, whose department of mathematics has been widely known, for the last three decades at least, as one of the most active strongholds of Algebraic Geometry. No member of this great school (with enough surplus to "export" some to the West) has ever dreamed, to be sure, of believing that s/he must be classified exclusively as "algebraist" rather than "geometer" in general.

A recent, fairly exhaustive, thorough, and no doubt very learned survey on "The Emergence of Princeton, 1896-1939"[60] somehow managed to skip the term "algebraic geometry" in toto throughout its presentation – a very serious oversight. As a matter of fact, the entire book that contains this survey as a part almost fails to mention[61] the term throughout its 387 pages (except once, in p. 287). Algebraic Geometry, usually defined as a "*geometric* theory of algebraic function fields in *n*-variables" (hence related to Number Theory via automorphic functions, indeterminate equations, etc.), has had as its prolific specialists, among many others, S. Lefschetz (of *L'analysis situs et géométrie algébrique*, 1924) and then, an equally famous leader in the area at issue, A. Weil (of *Foundations of Algebraic Geometry*, 1946), both at Princeton of course!

In the same book the contributor (a jack-of-all-trades of logician) on "Some Philosophical Remarks on Nineteenth-Century Transformation of Mathematics"[62] presents Dedekind's *Was sind und was sollen die Zahlen*? for the 1,001th time, and this, obviously without any inkling of Dedekind as one of the earliest pioneers in Algebraic Geometry. Such are the facts of "scholarly" life in the Academic Assembly Line.

<div style="text-align: right">Old Dominion University</div>

NOTES

1. Teiji Takagi, 1875-1960, a demi-god in the eyes of all contemporary mathematicians in Japan; until 1935, and even much later after his retirement from the (then) Imperial University of Tokyo, he was always at the center of mathematical creativity, surrounded by a galaxy of brilliant pupils (cf. nn. 2-3 below).
2. Shigekatu Kuroda, one year older than Iyanaga Shokichi (1906-) who eventually occupied the chair of Takagi, had to remain a "marginal" mathematician as long as he was known as an expert in mathematical logic at U. Tokyo. Only when his repertoire had grown much broader and greater, he was made a professor at the (then newly instituted, toward the end of WW II) Imperial U. of Nagoya.
3. Tongue-in-cheek? After all, Gödel had already entered the center stage, and Hilbert-Bernays' *Grundlagen der Mathematik* had seen its first volume by 1934. Nevertheless, this anecdote was (and still is?) well-known among his pupils, the spirit(?) of which was quite tangible if not explicitly visible

in his historical work: *Kindai–sugaku–shidan* (The Story from the History of Modern Mathematics). As for some derogatory remarks on "axiomatics" or "foundations of mathematics" in the West, they are too many to be noted here (some by F. Klein, A.N. Whitehead himself!).

4. Cf. his paper, "Mathematical Logic is neither Foundation nor Philosophy", in *Philosophica Mathematica*, II, v.1 [1986], pp. 3-14; this article was instantly translated into Italian and Russian.

5. Modo maxima rerum, tot generis natisque potens – unc trahor exul, inops. Neither F. Max Müller (1896) nor Norman Kemp Smith (1929) condescended to give his English translation except that, following the earlier commentaries by German Kantians, the exact location ("xiii, 508") in the *Metamorphoses* was noted. (After all, no student majoring in philosophy without a working knowledge of Latin and a smattering of Greek was even conceivable at European universities up to the 1960s, or perhaps even today?).

6. Not even Plato was quite ready to move from the individual "mathémata" (subjects to study, such as geometry, arithmetic, etc.) to "mathématiké" in the generic sense that became meaningful in much later centuries. (If a long and involved story is on demand, cf. J. Fang, *Sociology of Mathematics and Mathematicians*, Paideia, 1975, pp. 209-224 in particular, or the entirety of Chap. 5, pp. 149-224.)

7. The "mathémata" by the time of Plato did contain, even through not explicitly, much of the later Trivium and Quadrivium (cf. n.6 above).

8. A fairly large number of some glittering names may be readily compiled, consulting the obituaries of Forsyth, Whittaker et alii (or papers and a book on Whitehead by Victor Lowe, etc.).

9. Cf. J. Fang, "Kant as 'Mathematiker'," *Philosophia Mathematica*, II v.1 [1986], pp. 63-119. It seems immodest for me to cite my own paper, but in the spirit of "faut de mieux" (because even the most erudite German Kantians failed somehow or other to study "Kant als Mathematiker" in the light of the entire history of 17-19C. mathematics proper) such an immodesty may be forgiven here.

10. The self-acclaimed "Nachfolger" of Leibniz was weak in mathematics (perhaps incomparably so) if seen from the immense height of Leibniz himself; his "most extensive work" in mathematics was represented by a dictionary or such an expository textbook on basic mathematics as *Elementa matheseos universae* (in 4 vols. or its equivalent in German).

11. On this topic at least, the good old *History of Mathematics* by David Eugene Smith (Ginn, 1923, v.1, pp. 256, 337) is still the clearest and the most concise.

12. Especially via Madame du Châtelet's work? (cf. n. 9 above).

13. Cf. Pieranna Garavaso, "Wittgenstein's Philosophy of Mathematics: A Reply to Two Objects," *The Southern Journal of Philosophy*, v.26 [1988], pp. 179–191.

14. *The Foundations of Geometry* in 1902–translation in English, it was originally founded on the very ordinary–sounding title of his seminar in 1899: "Elemente der Euklidischen Geometrie" (noted down faithfully by H.v.Schapter, this note–form still exists); for detail in its historical background, cf. J.Fang, *Hilbert: Toward a Philosophy of Modern Mathematics*, II, Paideia 1970, Chap.5: "Metageometry" (pp. 82–93).

15. S.G. Shanker, *Wittgenstein and the Turning-Point in the Philosophy of Mathematics*, S.U.N.Y. Press, 1987.

16. Referring to Philip J. Davis – Reuben Hersh, *The Mathematical Experience*, 1981; the senior co–author had made himself known among working mathematicians by early in the 1960s via his widely acclaimed book on the theory of approximation.

17. Op.cit., p. 6

18. Referring to a New Math textbook (written by two full-professors of mathematics at U.Chicago and U.Wisc.-Madison) that allowed an operation between a set and its elements – cf. J. Fang, *Numbers Racket: Aftermath of New Math* , Kennikat Press 1967, Chap.3-4.

19. Cf.n.15. Unfair to single out Shanker, it must be emphasized here that the "blurred" vision has already been a firmly established tradition among most if not all of Wittgenstein-scholars, perhaps going all the way back to *The Nature of Mathematics* by Max Black or even earlier.

20. A mountain of books and papers have already amply commented on this problem; for the most recent, cf. "Ten Misconceptions about Mathematics and its History", by Michael J. Crowe, pp. 260-277, in *History and Philosophy of Mathematics* (Minnesota Studies in the Philosophy of Science, v.XI), edited by W. Aspray – P. Kitcher, U. Minn. Pr. 1988.

21. Shanker, *ibid.*, p. 210 (almost a propos of nothing).

22. Hardy, *Ramanujan*, Cambridge U. Pr. 1940 (later a Chelsea reprint).

23. Op. cit., p. 15 (italics by this writer).

24. Loc. cit. For a 18-page comment on Hardy's view, cf. J. Fang, *BOURBAKI: Toward a Philosophy of Modern Mathematics*, I,

Paideia 1970, pp. 109–126.

25. This oft-quoted (by Ayer, Quine et al.) aphorism appeared first in *Erkenntnis*, v. 3 [1932], p. 206.

26. Cf. n. 24 above; anybody with some knowledge in the history of mathematics must have his/her favorite example; cf. n. 20 above.

27. Referring to such [already?] "professional" geometers(!) as Archytas, Eudoxus et alii.

28. *Mathematics: The Loss of Certainty*, Oxford U. Pr. 1980. Both author and his many critiques (some well-known, such as Martin Gardner) seem to have made a federal case of "uncertainty" typically in an argumentum ad ignorantiam or plain equivocation. Under such circumstances as these, then, one may demand a more thorough and clearer philosophical analysis in advance.

29. Cf. n. 14 (of my own work, for the sake of brevity, in addition to such an admirably readable book by Constance Reid, appearing in the same year).

30. E.T.Bell, *Men of Mathematics*, 1937: "the fiercest frog-mouse battle (as Einstein once called it)".

31. In the manner, for instance, Dirac's delta-function had been a "puzzle" to mathematicians until the theory of distributions by Laurent Schwartz was able to clarify it; cf. *BOURBAKI* (n. 24), p. 139.

32. Cf. Bell *Development* (n.30), pp. 413–4. A much larger book by M. Kline, *Mathematical Thought from Ancient to Modern Times (Oxford, 1972)*, mentions neither Dirac's delta-function nor Heaviside's operator for whatever reason (even though his contempt for BOURBAKI was widely known).

33. Cf. his *Pragmatism*, 1907, p. 11 – very likely inspired by Fichte who wrote: "Was für eine Philosophie man wähle, hängt ... davon ab, was für ein Mensch man sei ..." (WW. I, 1, 434)?

34. G. Cantor, *Abhandlungen mathematischen und philosophischen Inhalts* (edited by E. Zermelo – A. Fraenkel, 1932; OLMS reprint, 1966), p. 182.

35. Loc. cit. This short line and this alone, out of context, is somehow or other the favorite of those who quote G. Cantor.

36. Since I have yet to see any paper or note at all related to this topic in English, I am referring here to the two articles in Japanese, both written by the young S. Iyanaga, included in this pre-WW II book in Japanese, *Junsui-sugaku no Sekai* (The World of Pure Mathematics), Kobundo (Tokyo), Japan, 1942, pp. 169–188. In his first short article (before he had actually read any of Bieberbach's papers) he condemned B.

for racism, but in the second long paper, after he had carefully studied them, agreed with B. for the possibility of certain classifications. All the time, however, he kept on hinting his personal friendship with some earlier Jewish members of Bourbaki (not yet so known) and his teacher, Emil Artin, and generally the "noble, beautiful friendship" beyond races and nationalities in the world of mathematics.

37. The controversay over IQ had brought forth many critiques of "Intelligence" (a hotbed for equivocation!), hence new theories of many-splendored (at least seven different types of) "talent" by Howard Gardner and others since c. 1983, but this topics is manifestly beyond the scope of the present paper.

38. Cf. n. 36. Instead of any outright accusation of racism (anti-Semitism in particular) against Bieberbach and others, and this, more often than not without ever reading their papers, one may as well take advantage of "Lern- und Lehrfreiheit" to study this kind of possibilities. After all, what is the Academic Freedom for!?

39. Cf. (to name but one, in this severely limited space) Chap. 3 under the same title of Ivar Ekeland, *Mathematics and the Unexpected*, U. Chicago Pr. 1988 (French original in 1984: *Le Calcul, l'Imprévu*).

40. The myth(!) of geomemtry's "Comeback" (when it had never been away or "dead" in the first place) has a cousin – a similar myth created by a historian (now at Princeton U.) who declared that the "Invariantentheorie" had been dead while, in reality, it was quite alive and kicking; cf. F.S. Fisher, "The Death of a Mathematical Theory: A Study in the Sociology of Knowledge," *Archive for History of Exact Sciences*, v. 3 [1966], which became another paper, with little improvement (how could a downright false theory be "improved"?) within a year, appearing in *European Journal of Sociology*, v. 8 [19657]. A green Ph. D. may be forgiven for his ignorance (and proportionately louder voice), but who were all the seasoned and learned referees of both prestigious journals?

41. Of. *De docta ignorantia*, 1440. (He, or his work, was never once mentioned throughout *A History of Western Philosophy*, 1946, by Bertrand Russell despite the fact that the author could have felt an intimate affinity toward N.C. for the latter's prolific uses of the infinite for theology and philosophy – odd!).

43. Cf. n. 37 above.

44. The first *Critique*, exclusively concerned with the Human

Faculty, did not (and no doubt needed not) bother itself
with the Godly, but since the latter must be what the former
cannot be (per definitionem), one may conclude this much:
God can never make any mistake in Godly Knowledge (omnis-
cient, omnipotent, etc., hence eternally free from any hu-
manly "parallaxes" between percepts and concepts). Compare
this notion with the flagrant (though unconscious and un-
criticized) use of Deux ex machina by A.J. Ayer to demolish
the synthetic a priori: "A being whose intellect was infinitely
powerful would take no interest in logic and mathematics."
This in turn was an English quotation from Hans Hahn,
"Logik, Mathematik und Naturerkennen" (appearing at the
end of Chap. 4 of *Language, Truth and Logic*, 1935/46).

45.   The first to notice this radically restructured nature of the
second edition of the first *Critique*, at least in printing, was
H. Amrhein's monograph in 1909: "Kants Lehre vom 'Bewusst-
sein Überhaupt', *Kant-Studien*, Ergängzungsheft 10. This
was made the most of in my paper for G. Schrader's seminar
on Kant (1949/50). A promise of publishing an enlarged
version was made in my *Kant-Interpretationen*, I   (Verlag,
Regensberg, 1967, p. ix) and, over two decades later, it is
still incomplete.

46   Knowing no better (faute de mieux, to the letter), I began to
toy with this notion since 1976, in *The Illusory Infinite: A
Theology of Mathematics* (Paideia), pp. 134, 252, 256, 286f.,
324f., 331.

47.   Any Belgian (or any specialist in algebraic topology) can
recognize him, but he shall remain here nameless since the
all too hasty opinion was expressed in his private letter to
me.

48.   According to the evaluation of "progress" by Samuel
Eilenberg; cf. *BOURBAKI* (n. 24), p. 110 et pass.

49.   If wrong, I begged to be rectified.

50.   *Philosophia Mathematica*, under my editorship, is to offer
$1,000 prize for just such a paper.

51.   At the International Congress of Logic, Methodology, and
Philosophy of Science (Amsterdam, 1967); cf. J. Fang, "What
is, and ought to be, the Philosophy of Mathematics?" *Phil.
Math.*, v. 4 [1967], pp. 71–75. The same question has of late
been raised afresh by P. Kitcher; cf. *Hist. & Phil. of Modern
Math.* (n. 20 above), p. 293.

52.   Cf. n. 39 above. The "Comeback" of geometry, in fact, was
first announced (to my best knowledge) by those devotees to
Mandelbrot's fractals – most loudly at any rate, over a
decade ago. Since then, of course, just about any book to

popularize the notion of the infinite (by Rudy Rucker et al.) never failed to show off its familiarity with the research front of Mathematics Today by adding a page or two, at times even a section, on fractals.

53. Cf. n. 40 above.

54. The relation, mathematical as well as academic, between Klein and Hilbert was not always smooth or routine; cf. J. Fang. *Hilbert*, Chap. 4.

55. Needless to add, in contrast to the "Three Big B's" (Bach, Beethoven, Brahms).

56. Such plain trios as 3-4-5, 5-12-13, 2-15-17(?) might have been numerically known, but without any notion of root-extraction (and with much difficulty even in simple divisions), the Pythagorean Theorem in antiquity was essentially qualitative; cf. D.E. Smith, *ibid.* (n. 11 above), v. 2, p. 288. For a more learned, and up-to-date, knowledge of his times, cf. Kenneth Sylvan Gutherie, *The Pythagorean Sourcebook and Library*, Phanes Press,1987. What a fabulous "progress" since (e.g.) E. Franck, *Plato und die sogennante Pythagoreer*, 1923, via W. Burkert, *Weisheit und Wissenschaft: Studien zu Pythagoras, Philolas und Platon*, 1962; C.J. Vogel, *Pythagoras and the Early Pythagoreanism*, 1966, etc. ("etc." is almost an insult to this area of so many profound and erudite classicists)!

57. Too many to name the current crop of researchers, only the illustrious pioneers may be briefly mentioned here: Dirichlet, Hermite, Minkowski, J.F. Koksma, E. Hlawka, H. Weyl, C.L. Siegel, K.F. Roth, S. Lang, ... (these names make it self-explanatory that this area of studies is inseparable from Diophantine approximations, D. geometry, etc.).

58. The (faulty) proof was reported at a Max Planck Institute meeting on this date, which was then made into an international news, even in such weeklies as *TIME* (publishing an essay, "The Joy of Math, or Fermat's Revenge," in 18.4.88 issue).

59. It cannot be emphasized too much (and repeated too often, as here) that, if written on the level of undergraduate mathematics alone, both history and philosophy of mathematics cannot help but face distortions and misinformations to the degree of surreal farce.

60. Included in *History and Philosophy of Mod. Math.*; (n. 20), by William Aspray.

61. Just once, like "... topology, algebraic geometry, ..."

62. By Howard Stein, possibly reading Klein's *Entwicklung der Mathematik* (in 19.C) mostly for anecdotes?