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THE PARADOX OF VOTING WITH INDIFFERENCE : Some Implications of Transitivity Assumptions

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Introduction

The paradox of voting, first discovered in the eighteenth century (see Black, 1958 and Riker, 1961 for discussions of historical background) resurfaced in the mid-twentieth century in seminal works by Black (1958) and Arrow (1951). They demonstrated that for an odd number of individuals and at least three alternatives in a society, there is no "democratic social welfare function" which can guarantee that transitive individual preference (and indifference) relations will generate a transitive relation for the entire society. While the result for the paradox of voting holds for both the preference and indifference relationships, the emphasis in discussions has been on the former virtually to the exclusion of the latter. In this paper, my concern will be with the effects of individual indifference on the paradox of voting in particular and the general question of aggregating individual preference-indifference orders into collective choices for a society. This will entail examining the logic of the paradox of voting under indifference, assumptions about the transitivity of the indifference relation, alternative conceptions of the meaning of indifference, and the question of how allowing individual indifference affects the probability of reaching a collective decision.

To discuss this, consider the argument behind the problem of a cyclical majority, the "end result" of the paradox of voting. First, consider only strong orders, i.e., only preference relations. The problem which a social welfare function is to resolve is the determination of a transitive social order among three or more alternatives by aggregating the preference orderings of three or more voters in such a way that the preference of no single voter (or outside dictator) *implies* a particular social choice. Individual preferences or profiles, are assumed to be transitive. Designate individual i's preference for alternative x over alternative y by xP_iy — and his preference for y over x by yP_ix. Further, assume that the relation is connected and reflexive : i.e., either xP_iy or yP_ix, but not both. A connected, reflexive, and transitive order for individual i over the alternative triple [x, y, z] such that xP_iy and yP_iz implies that xP_iz. The *profile* for i corresponding to the above preference ordering is designated as xyz.

Without placing any restrictions on admissable profiles, can a set of transitive individual preference orderings guarantee the generation of a "democratic" profile for an entire society (which may be as small as three members) which is itself transitive ? To see that it cannot, consider the following profiles for three individuals : *xyz*, *yzx*, and *zxy*. A majority of individuals prefers x to y; a majority prefers y to z. For a transitive social preference ordering, we thus require that a majority must prefer x to z. However, the profiles reveal just the opposite to be the case : A majority prefers z to x. Thus, the assumption of transitivity (at the level of the society) is violated and we have a cyclical pattern of dominance among the three alternatives.

The situation can easily be generalized to weak orders, i.e. relations which allow an individual either to prefer one alternative to another or to be indifferent between them. We denote this relation by xR_iy, which means that alternative x is viewed by individual i as at least as desirable as alternative y. I.e., $xR_iy = xP_iy$ *or* xI_iy, where I_i indicates that individual i is indifferent between a pair of alternatives. Thus, $xR_iy \rightarrow \sim yP_ix$. If we assume that the weak order categorized by xR_iy and yR_iz must be transitive for all i, then it is still possible to derive cycles such that zR_ix. Indeed, the possibility of a paradox of voting increases markedly because now we have many more possible orderings to consider. If we restrict our attention to strong orders, we can only derive six possible profiles for three alternatives, all of which are consistent with the assumption of the transitivity of individual preferences. When we consider weak orders, however, there are 27 possible outcomes, only 13 of which are consistent with individually transitive relations (for both preference and indifference.) These are listed in Table I. However, these 13 possible outcomes represent a more complicated situation than the six situations which might result from strong orders.

The problem of obtaining a social welfare function for individual weak orders is complicated by the large number of intransitivities which can be obtained for the three-valued alternative, on the one hand, and the possibility of a social choice which is itself indifferent

Table I

Transitive Individual Weak Orders*

xPy	yPz	xPz
xPy	yIz	xPz
xPy	zPy	zPx
xPy	zPy	xIz
xPy	zPy	xPz
xIy	yPz	xPz
xIy	yIz	xIz
xIy	zPy	zPx
yPx	yPz	zPx
yPx	yPz	xIz
yPx	yPz	xPz
yPx	yIz	zPx
yPx	zPy	zPx

*subscripts deleted

over two or three alternatives. Such intransitivities and societal indifferences seem*a priori* to reduce substantially the proportion of cases in which there is an unambiguous collective choice as the probability that an individual will be indifferent over any pair of alternatives increases. Restricting our criterion to a social decision function (in which we only require the aggregation mechanism to produce a single social choice instead of an entire transitive ranking), we can somewhat reduce the probability that no alternative will be uniquely preferred, and treating the collective indifference relation as intransitive will amount to employing a social desicion function to arrive at a collective decision when a cycle involving indifference occurs.

This paper will be to treat *individual* indifference as a transitive relation. The literature on social choice theories is divided on the question of how to treat individual indifference. There are persuasive arguments by Majumdar (1958) and Fishburn (1970, 1973), that the xI_iy (individual i is indifferent between alternatives x and y) relation should not be constrained to meet the assumption of transitivity. The logic behind these arguments is that there is a very fine line between an individual's decision to prefer x to y and a possible alternative decision to be indifferent between them. Specifically, the distinction may be so small that only a "just noticeable difference" (jnd) between two alternatives may yield a preference relation for one individual, whereas another member of society may not be able to draw such a distinction so easily. Thus, what may appear to be a relatively straightforward choice for some members of society may not be so easily made by others.

Nevertheless, I shall adopt the position that individual indifference relations should follow the assumption of transitivity. This posture shall be taken to simplify what shall nevertheless be a quite complex description of alternative social states if collective indifference is not required to be transitive. This paper will thus be concerned with the following problems : (1) what are the effects on social welfare functions if we assume that collective indifference relations are transitive ? ; (2) what are the effects on social decision functions if we drop the assumption of transitivity for collective indifference relations ? ; (3) how do we treat alternative interpretations of collective indifference (three of which shall be outlined below); and (4) to what extent does the probability of (transitive) individual indifference relations between pairs of alternatives affect the possibility of obtaining a unique social choice, on the one hand, or an undominated set of choices, on the other hand, under the varying assumptions of whether collective choices are assumed to be transitive for the indifference relation ? Throughout this paper, we shall assume that we have an impartial culture. I.e., $p_i^*(xP_iy) = p_i^*(yP_ix)$ *for all i*, where p_i^* is the probability of the event for all [x, y]. This is a relatively standard assumption in examinations of the paradox of voting, although it is quite restrictive. Again, the basic justification for employing an impartial culture is to simplify the model in the early stages of its development. Buckley and Westen (1974) have examined the probability of obtaining the paradox under partial cultures, but they do not allow for individual weak orders. The complexities of their model, which is itself a preliminary one, provide a note of caution in attempting to accomplish too many far-reaching results at once.

The evaluation of the probabilities for 25 values of $p_i(xI_iy)$ is obtained through a computer simulation initially designed by Uslaner (1974) and refined with Richard G. Smith (1974) both computationally and conceptually[1]. I.e., the earlier paper underestimated the frequency of the paradox of voting for large values of P_i and also did not even attempt to distinguish among alternative meanings of the results for collective indifference. In this paper, we shall consider societies as small as three members and as large as 75 (which is a reasonable asymptote for results corresponding to an infinitely large population). Only three issue alternatives will be examined. The discussion of the simulation routine will be presented below. First, we should address the central conceptual problems which arise when: (1) we consider social welfare functions under situations involving weak orders; (2) the interpretation of indifference is allowed to take on different meanings; and (3) the assumption of transitivity for collective choices is relaxed.

The Paradox of Voting and Individual Indifference

The 13 orders for a society when individual indifference is allowed to occur but not to violate transitivity produce a set of possible outcomes which can be categorized into seven distinct categories. In this section of the paper, I shall discuss each of the possible categories in theoretical terms; the next section will present the simulation results. The theoretical discussion is of importance because it will introduce some complications into social choice theory which have not been discussed within a single framework in the previous literature, particularly under alternative conceptions of what the collective indifference relation means. Before entering such a discussion, however, it will be useful to demarcate the possible alternative meanings of collective indifference.

Spatial models of party competition have considered two alternative conditions for abstension: (1) indifference among a set of candidates and (2) alienation of the voter from the set of candidates (Hinich *et al.*, 1972), Hinich and Ordeshook, 1970). The interpretation of indifference in more general social choice situations as comparable to abstention in elections is consistent with the approach of spatial models since the term "indifference" has been used quite loosely throughout the literature. Spatial models distinguish between indifference and alienation in that the former term indicates that the voter simply does not prefer one candidate to another. Alienation, on the other hand, occurs when the voter's ideal point is at such a distance from the position(s) taken by the various candidates that there is no benefit seen by voting at all. Spatial modeling terminology thus permits us to draw the following distinction: (1) "indifference"-based abstention may be considered to represent a situation in which all alternatives are viewed as (relatively) equally desirable; while (2) "alienation"-based abstention can be considered to represent a situation in which all alternatives are viewed as (again, relatively) equally distasteful. In electoral systems which permit the voter to cast a blank ballot (such as the French), one can attempt to distinguish between "indifference" and "alienation" by measuring the latter as a function of the number of blank ballots cast (Rosenthal, 1975)[2]. The more generalized literature on social choice theory does not make such a distinction, however.

The indifference relation, xI_iy for individual i or xIy for a society, does not distinguish between equally preferred and equally detested alternatives. As such, it does not consider the possibility that individuals who find the various alternatives equally attractive might be considered to have "abstained" in the voting process when the aggregation of individual choices into a collective decision is formulated. Such abstentions may then result in a greater probability of obtaining a social choice than if such "indifferent" voters had their ballots counted equally to those who have strong preferences. This treatment of the indifference relationship clearly does not require it to satisfy transitivity for the collective social choice. If, on the other hand, we treat abstention as based upon alienation, we should consider indifference in more general social choice theory to reflect a distaste on the part of individuals (and possibly the entire collectivity) for some or all of the alternatives posed, even though there may be some alternatives which are pairwise preferred to others. In such situations, we want to insist on the transitivity of the social preference ordering (since "blank ballots" are considered legitimate in electoral systems which provide for them), thereby increasing the probability that a collective decision will not uniquely prefer one alternative to all others.

There is, however, a third possible interpretation of societal indifference. Let n(aRb) denote the number of people in a society who either prefer a to b, or are indifferent between them (given either interpretation of indifference discussed above). Then, suppose that : n(aPb) = n(bPa) or n(aPb) = n(aIb), where [a, b] represent any and all possible pairwise combinations of the triple x, y, z. We denote either of these relationships as aEb : I.e., the number of people who prefer alternative a to alternative b is equal to the number of people who prefer (or are indifferent between) b to a. May (1952) and Pattanaik (1975) treat such relationships as equivalent to the regular indifference relation. However, it is not clear that such an interpretation is desirable. Under the first interpretation of the indifference relation, we assumed that for at least some pair of alternatives, the various members of a society consider alternatives a and b equally desirable; the second interpretation finds a majority on some pairwise comparison finding a and b equally repugnant. Now the aEb relation, particularly as it relates to strong orders, says something quite different : There is no majority indifference (or preference), but rather an equal number of members of society prefer a to b as prefer b to a. For each subgroup, (aP_gb) and (bP_ha) — where g and h are subgroups of the total population, the degree of intensity for the preference may be quite strong.

Even extending this logic to weak orders requiring transitive individual indifference relations, it is clear that there are varying degrees of collective indifference. Indeed, the aEb relation can be considered to represent indifference only in the sense that it does not provide a basis for determining a *preference* relation among a set of alternatives. Otherwise, it seems to be quite different from the other meanings of indifference we have considered. The aEb relationship, unlike aIb, is clearly required to follow the transitivity condition for collective choice. We shall see below that this leads to a situation in which the probability of obtaining a consistent social decision function is decreased, although this occurs through the indifference relation rather than the more traditional paradox of voting. The interpretation of such an intransitivity as societal indifference is chosen because it seems the most plausible, on the one hand; on the other hand, it is interesting to note that intransitivities can produce situations in which the indifference relationship seems to be more justifiable than the paradox.

Having discussed the various meanings of indifference, I turn now to a discussion of the seven categories of outcomes which can result from the 13 possible weak orders which satisfy individual transitivity for both preference and indifference. In the discussion below, the aIb notation will be employed for both of the first two meanings of indifference. The possible outcomes are :

(1) *Choice :* aPb, bPc, aPc. This, of course, is the familiar result which will be obtained if we assume only strong individual orders. It is obviously a potential outcome even allowing for individual indifference.

(2) *Strong Indifference :* aIb, bIc, aIc. There is no societal choice regardless of transitivity assumptions Each alternative is equally desirable or repugnant to the others by pairwise comparisons, at least as measured on an ordinal scale.

(3) *Weak Indifference :* aPc, bPc, aIb. There is a tie for first place between alternatives a and b in the sense that the indifference relation does not allow us to discriminate between a and b, both of which are preferred to c. No violation of the transitivity of the I relationship is involved, even if both a and b are considered undesirable by a majority of voters. Each alternative is still preferred to alternative c.

(4) *Moderate Indifference :* Either (1) aEb, bEc, aEc; or (2) aEb, bEc, aIc. In the first instance, these is a violation of transitivity and

society is indifferent among the three alternatives but not, as in strong indifference, because aIb, etc. is the social choice. Rather, whatever cycles there are affect each outcome equally and reflect an even balance between support for various alternatives. The second case of moderate indifference is a more sharply drawn instance of intransitivity, since it involves the I relation explicitly (whereas it is possible not to use the I relation in the first case); however, the overall result is the same.

(5) *Pseudo-Weak Indifference :* aPb, bPc, aE*c or cE*a, where E* is the E relation involving n(aPc) = n(aIc) such that n(aPc) > n(cPa). Whereas weak indifference does not violate the transitivity of the I relation for a collective choice function, pseudo-weak indifference does. If we do not require the transitivity assumption to hold, then society's most preferred outcome is alternative a. Thus, what appears to be an instance of weak indifference (if one adopts the May-Pattanaik view of indifference stated above) is either a violation of transitivity in which the I relation is the societal result for the same reasons as given for moderate indifference or there is a social choice if the transitivity assumption is dropped.

(6) *Pseudo-Paradox :* aRb and aRc such that it is not the case that *both* aPb and aPc. This is what Klahr (1966) calls a "type 1 paradox," i.e., there is a preferred alternative from the triple [a, b, c], but there is a cycle among at least (n—1) of the n alternatives (only n—1 if strong individual orders are required; the cycle may be complete for weak orders depending upon whether the assumption of collective transitivity is assumed). For this analysis, the pseudo-paradox will be considered to be a paradox of voting for transitive I relations, but will yield a social decision function if I is not transitive.[3]

(7) *Paradox :* All remaining outcomes. By process of elimination, all of these cycle to yield what Klahr calls a "type 2 paradox", i.e., the classical cyclical majority.

Thus, even assuming that the individual indifference relation must satisfy the property of transitivity, we have a much more complex situation than in analyses which only consider strong orders. The question to be posed next is : What are the implications of allowing individual indifference (under alternative meanings) to enter into calculations about the possibility of reaching a choice for a collectivity ? Tullock (1967 : 37-49) has argued that the paradox of voting is not a very important problem from the perspective of social choice theory since the probability of its' occurence is not high. However, allowing for individual indifference complicates the problem in two ways : (1) there are cycles involving the I-relation;

and (2) even if we do not consider cycles involving the I-relation, the possibility that one or more of the types of indifference may prevent the attainment of a collective choice has potentially disturbing consequences. While any of the four possible indifference relations may not be normatively disturbing to some observers who are content as long as the selection process for an alternative is itself nonarbitrary, others may find the consequences of no societal choice due to one of the indifference relations to be either : (1) disturbing, in that no choice is generated by the social welfare function when the function is adopted to provide such a selection; or (2) distasteful, in that whatever choice may be generated by a society may be *strongly* opposed by a large enough segment of society to deny that alternative a majority or plurality either under a social welfare function or even a social decision function. In any event, the problem of reaching a collective decision is certainly complicated by the introduction of individual indifference into the model and comparisons of the results for strong orders are certainly important. It is to this problem that I now turn.

Estimating Societal Choices with Indifference

How frequent is the paradox of voting ? For impartial cultures with varying sizes of society and number of alternatives, direct mathematical estimates of the paradox have been provided by Black (1958), G. Th. Guilbaud (1952), Niemi and Weisberg (1968), Garman and Kamien (1968), DeMeyer and Plott (1970), May (1971), and Hansen and Prince (1973). Monte Carlo approaches have been employed by Campbell and Tullock (1973), Klahr (1966), and Pomeranz and Weil (1970). In general, the simulation results appear to approximate the analytic results very well. Direct mathematical solutions are to be preferred since they are exact, but the complexity of the mathematical formulae sometimes makes this approach infeasible. For the present situation, even with all of the simplifying assumptions employed, this si certainly the case. For the 13 possible outcomes obtainable by assuming that individual indifference is a transitive relation, we must consider a multinominal expansion which not only requires knowledge of the probability that a given outcome will occur (which is not problematic), but also which is constrained by a set of equations which require that the probabilities of each collective outcome reflect the probabilities of individual decisions. This is perhaps the most complex of the problems which make a direct solution intractable.

Thus, a Monte Carlo method is adopted to derive estimates of the various collective outcomes discussed above. We consider a three-alternative situation. The simulation allows p_i, the probability of individual indifference to vary from zero to unity; 25 values of p_i are employed in the simulation (1.000 being omitted, since it will always, by definition, yield strong indifference) and these are listed in Table II below. The various values were selected so that the tails of the distribution could be examined in particular depth, since it is at these extreme values that variations in the results are the most interesting (as well as seemingly the most likely to produce more than incremental changes).

Table II

Values of the Probability of Individual Indifference Used in the Simulation

.000
.001
.010
.025
.050
.100
.150
.200
.250
.300
.333
.400
.450
.500
.550
.600
.667
.700
.750
.800
.850
.900
.950
.975
.990

The simulation routine developed has each individual "select" a profile from the 13 transitive preference orderings through calculation of the probability that the profile in question will actually be drawn and comparison of that probability (actually, the cumulative probability of the j^{th} profile, where $j = 1, 13$) with a figure drawn from a uniform random number generator. The simulation can handle impartial as well as partial cultures, but here the probability calculations become somewhat more complex. For an impartial culture, the probability that any of the 13 individually transitive orders will be selected can be readily determined (Uslaner, 1974 : 43; Smith, 1974 : 4). First, consider only those profiles in which the indifference relation is not found (profiles 1, 3, 5, 9, 11, and 13 in Table I). Let p_i be the probability of individual indifference for individual i for any dyad of alternatives, subject to the constraint of an impartial culture and let $q_i = (1 - p_i)/2$. Then, for these outcomes, we have the following probability for each possible profile :

$$(q_i^2 + 2q_i^3)/3.$$

For the profile in which an individual is indifferent over the entire set of alternatives (number 7 in Table I), the probability of selecting that outcome for an impartial culture is simply p_i^2. Finally, for the remaining ballots, we have the probability :

$$(2p_iq_i + p_iq_i^2)/3.$$

Before discussing the simulation results, it will be instructive to compare the computed probabilities of the paradox of voting in this study with both analytic and other Monte Carlo simulations. These are presented in Table III for societies of three, five, seven, nine, and eleven members. A sample size of 1000 was employed for this study, so that the results are only significant to one decimal point. The limitation of the number of "societies" considered was dictated largely by the size of the problem in comparison to other results which only consider the probability of the paradox with $p_i = .000$. Indeed, these are the only comparisons which can be made. This sample size is the same as that of Campbell and Tullock (1963), but Pomeranz and Weil (1970) employed samples of 15,000 while Klahr (1966) utilized 10,000 societies. The results in Table III indicate that the simulation performs better than the Campbell-Tullock one except for seven-member societies where the present results underestimate the probability of the paradox and the Campbell-Tullock simulation overestimates this probability. Except for this aberrant case, the simulation performs at least as well as the others (and better in some cases) than do the ones with larger samples. The study by Uslaner (1974), based upon a sample of 100,000 societies, is identical in design to the present work for $p_i = 0$ and the results for that simulation at the .05 confidence level are within .003 of the population values. For these results, even the seven-member society is predicted remarkably well in correspondence with the analytic results. Hence, the simulation appears to provide very satisfactory results. I turn now to an examination of the various possible outcomes for the 25 values of p_i selected.

Table III

Comparison of Simulated and Analytically Derived Values of the Paradox of Voting and Probability of Individual Indifference Equal to Zero

	Analytic Results			Simulated Results				
Number of Members in Society	Garman and Kamien	DeMeyer and Plott	Niemi and Weisberg	Present Work	Uslaner	Klahr	Campbell and Tullock	Pomeranz and Weil
3	.05556	.05556	.0556	.055	.05407	.0555	.057	.05
5	.06944	.06945	.0694	.072	.06790	.07	.078	.07
7	.07502	.07510	.0750	.066	.07475	.08	.084	.07
9	.07798	.07798	.0780	.078	.07793	*	.089	.07
11	.07981	.07981	.0798	.077	.08040	*	.085	.08

*Value not reported

Results of the Analysis

In this section, we shall consider the results of simulations for societies composed of 3, 5, 9, 25 and 75 members under the assumptions of both transitive and intransitive collective indifference and under the various meanings of "indifference". Five societies with different numbers of members have been selected from a larger set for which results have been obtained to highlight some general trends. The overall results of the analysis are presented in Table IV. Table V presents the probabilities obtained by treating the I relation for a society as transitive; Table VI presents similar probabilities for as intransitive I relation in which pseudo-weak indifference and pseudo-paradoxes are assumed to yield a collective choice[3] but moderate indifference is treated as a separate outcome which is assumed to yield collective indifference as the societal result. The probability of an intransitive outcome is thus simply the probability of a cycle when the probability of individual indifference equals zero. Thus, Table V considers the I relation to reflect equal *dissatisfaction* with the varying alternatives which may be considered to be nonimposed; Table VI also takes into account the potential variations in individual choices which correspond to "indifference" in spatial models but considers ties between alternatives to be fundamentally different from such indifference.

For a three-member society, the probability of a collective choice is monotonically decreasing once p_i becomes non-zero. The sharpest rate of decrease occurs in the interval of (.333, .55), which we shall discuss in further detail below. On the other hand, strong indifference is virtually nonexistent for small values of p_i, although it begins a sharp increase for moderate values and is the most frequent result after $p_i = .60$. Weak indifference increases up to $p_i = (.25, .30)$ and decreases monotonically thereafter. Moderate indifference, like its weak counterpart, also has a curvilinear relationship to p_i and is not a dominant outcome under any conditions. Pseudo-weak indifference is not found at all for a three-member society, which is understandable given the very low probability of a tied result for such a small group. The paradox, interestingly, has another curvilinear relationship with p_i, peaking at .45. Note that the curvilinear results tend to peak at similar points; this will become even more evident for larger societies. It will be discussed below in the light of an analytic result due to Mahoney (1974).

The probability of a choice for a five-member society has a similar functional form to that for a three-member society, except that the slope of the downward trend becomes larger at a lower value of p_i

Table IV

	3-Member Society						5-Member Society					
P_i	Choice	Strong Indiff	Weak Indiff	Mod Indiff	Pseudo Weak	Para-dox	Choice	Strong Indiff	Weak Indiff	Mod Indiff	Pseudo Weak	Para-dox
0	.945	.000	.000	.000	.000	.055	.928	.000	.000	.000	.000	.072
.001	.947	.000	.003	.000	.000	.049	.928	.000	.003	.000	.000	.069
.01	.920	.000	.020	.000	.000	.060	.905	.000	.015	.000	.001	.079
.025	.917	.000	.035	.000	.000	.048	.854	.000	.039	.000	.004	.103
.05	.847	.000	.073	.000	.000	.080	.792	.000	.064	.002	.012	.130
.10	.769	.000	.113	.010	.000	.108	.685	.000	.115	.011	.029	.160
.15	.686	.000	.157	.019	.000	.138	.620	.000	.113	.028	.060	.179
.20	.632	.005	.150	.040	.000	.173	.534	.000	.107	.058	.084	.217
.25	.523	.016	.208	.070	.000	.183	.421	.003	.123	.090	.079	.284
.30	.457	.020	.206	.083	.000	.234	.348	.010	.104	.138	.092	.308
.333	.393	.045	.200	.099	.000	.263	.310	.011	.111	.178	.074	.316
.40	.361	.070	.167	.150	.000	.252	.225	.038	.084	.264	.068	.321
.45	.286	.103	.180	.150	.000	.281	.206	.069	.096	.275	.042	.312
.50	.250	.158	.173	.199	.000	.220	.143	.138	.056	.344	.038	.281
.55	.188	.216	.146	.218	.000	.232	.108	.202	.066	.354	.028	.242
.60	.151	.302	.134	.204	.000	.209	.068	.291	.044	.354	.018	.225
.667	.093	.405	.104	.198	.000	.200	.048	.440	.029	.315	.013	.155
.700	.085	.481	.076	.184	.000	.174	.037	.535	.026	.258	.005	.139
.75	.059	.603	.047	.156	.000	.135	.019	.682	.014	.184	.000	.101
.80	.045	.711	.025	.130	.000	.089	.014	.778	.012	.141	.000	.055
.85	.023	.821	.031	.063	.000	.062	.008	.870	.002	.082	.000	.038
.90	.014	.903	.012	.042	.000	.029	.000	.962	.001	.023	.000	.014
.95	.002	.975	.001	.015	.000	.007	.000	.997	.001	.002	.000	.000
.975	.001	.993	.000	.003	.000	.003	.000	.999	.000	.001	.000	.000
.99	.000	.999	.000	.000	.000	.001	.000	1.000	.000	.000	.000	.000

TABLE IV (CONTINUED)

	25-Member Society						75-Member Society					
P_i	Choice	Strong Indiff	Weak Indiff	Mod Indiff	Pseudo-Weak	Paradox	Choice	Strong Indiff	Weak Indiff	Mod Indiff	Pseudo-Weak	Paradox
0	.938	.000	.000	.000	.000	.062	.924	.000	.000	.000	.000	.076
.001	.929	.000	.000	.000	.000	.071	.897	.000	.006	.000	.000	.097
.01	.864	.000	.021	.000	.000	.115	.852	.000	.033	.000	.000	.115
.025	.829	.000	.042	.000	.000	.129	.857	.000	.049	.000	.000	.094
.05	.801	.000	.064	.001	.000	.134	.847	.000	.038	.000	.000	.115
.1	.814	.000	.074	.001	.000	.111	.850	.000	.051	.000	.000	.099
.15	.808	.000	.076	.003	.000	.113	.855	.000	.043	.000	.000	.102
.20	.799	.000	.089	.001	.003	.108	.868	.000	.053	.000	.000	.079
.25	.777	.000	.079	.002	.020	.122	.861	.000	.046	.000	.000	.092
.30	.687	.001	.098	.006	.017	.191	.815	.001	.069	.003	.001	.105
.333	.587	.011	.123	.016	.035	.228	.739	.003	.095	.002	.007	.146
.40	.311	.065	.128	.070	.029	.397	.307	.088	.128	.057	.015	.394
.45	.167	.189	.104	.103	.028	.409	.108	.374	.059	.081	.026	.374
.50	.087	.391	.051	.134	.013	.324	.018	.727	.021	.054	.004	.180
.55	.034	.601	.033	.102	.001	.229	.000	.942	.000	.017	.000	.041
.60	.010	.747	.008	.072	.001	.112	.000	.995	.000	.005	.000	.000
.667	.000	.955	.001	.023	.000	.021	.000	1.000	.000	.000	.000	.000
.70	.000	.985	.000	.009	.000	.006	.000	1.000	.000	.000	.000	.000
.75	.000	.996	.000	.004	.000	.000	.000	1.000	.000	.000	.000	.000
.80	.000	1.000	.000	.000	.000	.000	.000	1.000	.000	.000	.000	.000
.85	.000	1.000	.000	.000	.000	.000	.000	1.000	.000	.000	.000	.000
.90	.000	1.000	.000	.000	.000	.000	.000	1.000	.000	.000	.000	.000
.95	.000	1.000	.000	.000	.000	.000	.000	1.000	.000	.000	.000	.000
.975	.000	1.000	.000	.000	.000	.000	.000	1.000	.000	.000	.000	.000
.99	.000	1.000	.000	.000	.000	.000	.000	1.000	.000	.000	.000	.000

TABLE IV (CONTINUED)

9-Member Society

P_i	Choice	Strong Indifference	Weak Indifference	Moderate Indifference	Pseudo-Weak	Paradox
0	.922	.000	.000	.000	.000	.078
.001	.922	.000	.002	.000	.000	.076
.01	.889	.000	.027	.000	.000	.084
.025	.873	.000	.041	.000	.000	.086
.05	.817	.000	.058	.000	.000	.125
.10	.780	.000	.094	.000	.000	.126
.15	.723	.000	.125	.004	.002	.146
.20	.718	.000	.109	.006	.008	.159
.25	.602	.003	.157	.023	.011	.204
.30	.528	.006	.161	.023	.020	.262
.333	.441	.020	.156	.051	.019	.313
.40	.318	.057	.139	.112	.018	.356
.45	.246	.132	.117	.133	.019	.353
.50	.152	.207	.094	.167	.010	.370
.55	.111	.306	.064	.191	.008	.320
.60	.061	.451	.035	.181	.004	.264
.667	.018	.638	.019	.154	.002	.169
.700	.014	.733	.006	.128	.000	.119
.75	.004	.857	.002	.087	.001	.053
.80	.000	.941	.001	.040	.000	.018
.85	.000	.978	.001	.016	.000	.005
.90	.000	.995	.000	.004	.000	.001
.95	.000	1.000	.000	.000	.000	.000
.975	.000	1.000	.000	.000	.000	.000
.99	.000	1.000	.000	.000	.000	.000

Table V

P_I	3-Member Society		5-Member Society		9-Member Society		25-Member Society		75-Member Society	
	Trans Indiff	Intrans	Trans Indiff	Intrans	Trans Indiff	Intrans	Trans Indiff	Intrans	Trans Indiff	Intrans
0	.000	.055	.000	.072	.000	.078	.000	.062	.000	.076
.001	.003	.049	.003	.069	.002	.076	.000	.071	.006	.097
.01	.020	.060	.015	.080	.027	.084	.021	.115	.033	.115
.025	.035	.048	.039	.107	.041	.086	.042	.129	.049	.094
.05	.073	.080	.064	.144	.058	.125	.064	.135	.038	.115
.10	.113	.118	.115	.200	.094	.126	.074	.112	.051	.099
.15	.157	.157	.113	.267	.125	.152	.076	.116	.043	.102
.20	.155	.213	.107	.359	.109	.173	.089	.112	.053	.079
.25	.224	.253	.126	.453	.160	.238	.079	.144	.046	.092
.30	.226	.317	.114	.538	.167	.305	.099	.214	.070	.109
.333	.245	.362	.122	.568	.176	.383	.134	.279	.098	.155
.40	.233	.402	.122	.653	.196	.486	.193	.496	.216	.466
.45	.283	.431	.165	.667	.249	.505	.193	.540	.433	.481
.50	.331	.419	.194	.663	.301	.547	.442	.471	.748	.238
.55	.362	.450	.268	.624	.370	.519	.634	.332	.942	.058
.60	.436	.413	.335	.597	.486	.449	.805	.185	.995	.005
.667	.509	.398	.468	.483	.657	.329	.956	.044	1.000	.000
.70	.556	.358	.561	.402	.739	.247	.985	.015	1.000	.000
.75	.650	.291	.696	.285	.859	.141	.996	.004	1.000	.000
.80	.736	.219	.790	.196	.942	.058	1.000	.000	1.000	.000
.85	.852	.125	.872	.120	.979	.021	1.000	.000	1.000	.000
.90	.915	.071	.963	.037	.995	.005	1.000	.000	1.000	.000
.95	.976	.022	.998	.002	1.000	.000	1.000	.000	1.000	.000
.975	.993	.006	.999	.001	1.000	.000	1.000	.000	1.000	.000
.99	.991	.001	1.000	.000	1.000	.000	1.000	.000	1.000	.000

Trans Indiff = Transitive Indifference
Intrans = Intransitivity

Table VI

p_i	3-Member Society Choice	3-Member Society Indiff	5-Member Society Choice	5-Member Society Indiff	9-Member Society Choice	9-Member Society Indiff	25-Member Society Choice	25-Member Society Indiff	75-Member Society Choice	75-Member Society Indiff
0	.945	.000	.928	.000	.922	.000	.938	.000	.924	.000
.001	.947	.003	.928	.003	.924	.002	.936	.000	.918	.006
.01	.925	.020	.913	.015	.895	.027	.917	.021	.891	.033
.025	.917	.035	.889	.039	.881	.041	.896	.042	.875	.049
.05	.872	.073	.862	.066	.864	.058	.873	.065	.886	.038
.10	.822	.123	.802	.126	.828	.094	.863	.075	.873	.051
.15	.769	.176	.782	.141	.793	.129	.859	.079	.881	.043
.20	.750	.190	.763	.165	.807	.125	.848	.090	.871	.053
.25	.651	.294	.712	.216	.739	.183	.857	.081	.876	.046
.30	.636	.309	.676	.252	.732	.190	.833	.105	.845	.073
.333	.601	.344	.628	.300	.695	.227	.788	.150	.816	.100
.40	.558	.387	.542	.386	.614	.308	.675	.263	.640	.273
.45	.512	.433	.488	.440	.540	.382	.542	.396	.432	.512
.50	.415	.530	.390	.538	.454	.468	.362	.576	.126	.796
.55	.365	.580	.306	.622	.361	.561	.202	.736	.000	.959
.60	.305	.640	.239	.689	.251	.667	.061	.877	.000	1.000
.667	.238	.707	.144	.832	.111	.811	.000	.979	.000	1.000
.70	.204	.741	.109	.819	.055	.867	.000	.994	.000	1.000
.75	.139	.806	.048	.880	.005	.946	.000	1.000	.000	1.000
.80	.079	.866	.014	.931	.000	.982	.000	1.000	.000	1.000
.85	.030	.915	.008	.954	.000	.995	.000	1.000	.000	1.000
.90	.014	.957	.000	.986	.000	.999	.000	1.000	.000	1.000
.95	.002	.991	.000	1.000	.000	1.000	.000	1.000	.000	1.000
.975	.001	.996	.000	1.000	.000	1.000	.000	1.000	.000	1.000
.99	.000	.999	.000	1.000	.000	1.000	.000	1.000	.000	1.000

and the overall descent is much steeper. On the other hand, strong indifference displays a similar tendency but in the opposite direction : It has similar probabilities for lower values of p_i, but the rate of increase is greater than for three-member societies and the strong I relation more quickly dominates all other outcomes. Weak indifference peaks at the same p_i as for three member societies, although it seems to be a less likely outcome overall. Moderate indifference increases sharply from the results of a three-member society and peaks considerably later (.55 to .60). A five-member society seems considerably more likely to generate tied results for the various dyads than does a three-member one and, as we shall see, for larger societies as well. This result is not as anamalous as one might think : The larger the society, the lower the probability that any two alternatives in a triple such as [aPc, aIb, bPc] will be exactly tied. Given the transitivity of individual profiles, we can effectively rule out the possibility that *all* possible relations will be tied, as in one of the variants on moderate indifference, for a three-member society, but not at all for one with five voters. Pseudo-weak indifference also peaks in the vicinity of .33, but here the pattern is somewhat erratic (due in part to the sample size.) Finally the paradox is more frequent for a society of five voters that it is for one of three, a not unexpected result, given the deterministic solution; *but*, once the probability of a cycle begins to decline (p_i in the vicinity of .40 to .45.) the rate of descent is greater for the larger society. What is obviously occurring is that, as the size of the society becomes larger, the various indifference relations (and particularly strong indifference) tend to increase more strongly with p_i, thus lowering probabilities both of a collective choice and a paradox.

For a nine-member society, we find the rate of decrease for a collective choice to be smaller that for smaller groups at low to moderate values of p_i, but much greater at an beyond .50. Indeed, the probability of a transitive social welfare function is zero at p_i = .800 and less than .010 for p_i = .60. Strong indifference remains negligible for small values of p_i, but begins a rapid increase in the neighborhood of .45. Both weak and moderate indifference have curvilinear relationships with p_i, the former peaking in the range of (.25, .333) and the latter at .55. Pseudo-weak I is not very probable at all, but also has a skew-symmetric distribution which is for a given interval, unimodal. Finally, the paradox has a similar distribution, but at p_i = .50 reaches the largest values we have encountered so far in Table IV : .370. It is obviously the case that as the number of alternatives increases, so does the probability of an intransitivity. However, subject to domination by the various I relations when p_i is sufficiently large, cycles appear to be considerably more probable when indifference is entered into the problem. This is hardly surprising since the number of possible cycles has been increased.

Consider together, now, the results for societies of 25 and 75 members. Both are large enough so that the results for strong orders very closely approximate the asymptotic derivations for infinitely large samples. In each case, we see that the probability of a collective choice is much higher than observed for smaller samples, even for moderate values of p_i. However in the interval (.333, .40) there is a precipitous drop in the probability of a collective choice in a step-function fashion and this probability rapidly decreases to zero for values of p_i which themselves do not seem very high. While strong indifference is virutally non-existent for individual probabilities of indifference in which there is likely to be a collective choice, there is a similar step-function increase in the strong I probabilities at p_i = .50 for a 25 member society and p_i = .45 for 75-member societies. The rise of the probabilities once again occurs in the vicinity of p_i = .40. Weak and moderate indifference both have curvilinear relations with p_i, and for both 25 and 75 member societies the effects of each type of indifference are considerably dampened in comparison with smaller groups of voters. In particular, both types of indifference seem to decrease at approximately the same points that strong indifference becomes the dominant outcome. Pseudo-weak indifference is much less important in either society. While never even attaining the ten percent level for any value of p_i and any number of voters considered (the maximum is .092 for p_i = .30 in a five-member society), the logical problem raised by this type of relationship cannot simply be waived away. Fortunately, for the theory of collective choice, the magnitude of the problem is not is not great, particularly in societies with relatively large numbers of voters. The paradox has skew-symmetric unimodal distributions for both societies, with distributingly large values: approximately 40 percent of all outcomes yield cycles in the (.40, .45) range for the two larger societies. For some values of the probability of individual indifference, then, the paradox of voting does pose a formidable problem for collective decision-making if the I relation for a society is transitive. As strong indifference overwhelms the paradox and social choices, however, the problem dissolves for sufficiently large values of p_i these "sufficiently large" values are not so very much greater that those for which the strong I relation holds, on the one hand, or not too much smaller than those for which these is a social welfare function satisfying transitivity for reasonably large societies.

These results indicate that there are three primary results for the paradox of voting with transitive individual and collective indifference and reasonably large societies : a social choice, a cyclical majority, or strong indifference. The remaining types of indifference are not critical for large societies for virtually any value of the probability of individual indifference. A fascinating pattern has emerged, however. When we observe turning points in curvilinear relations or, in particular, sharp increases or decreases in slopes (in a step-function fashion), these points tend to occur in the vicinity of $p_i = .40$. Indeed, as the societies under consideration become larger, such points tend to cluster more closely around this value than they do for smaller groups. What is the significance of this finding ? Substantively, there is nothing which appears intuitively appealing about .40 as a critical point. However, the importance of these Monte Carlo findings is highlighted by an analytical result in an unpublished work by Mahoney (1974 : 7), which establishes that for an impartial culture and an arbitrarily (i.e., virtually infinite) large society, the value of p_i needed to *ensure* (i.e., with a probability of unity) that the societal result will be strong indifference is .38829. Note that this result holds only asymptotically and for impartial cultures in which the indifference relation is treated as transitive (although the estimation of the probability involves the possibility that the 14 outcomes which are intransitive of the total of 27 will occur).

However, the import of the result, at least from the results presented here, goes beyong that of what Mahoney demonstrated. In particular, it appears that as p_i approaches the critical value, the paradox of voting will become more important and the probability of a choice will begin to decline. Both will, drop precipitously as the critical point begins to be reached and only societal indifference will occur at and after that point. Mahoney did not make the distinction between the probability of the paradox occurring and that of a social choice because his analysis, based on impartial cultures, treated the probability that any particular outcome will occur solely as a function of the likelihoods of given profiles. No attempt was made to distinguish the behavior of any profiles other than strong indifference and other strong orders (i.e., not including the indifference relation).

If we consider the I relation to be transitive from the perspective of collective decisions, we note that : (1) except for very small societies, the intransitivies which may arise are much more likely to be a function of the paradox than of moderate indifference; and (2) the results presented in Table V indicate that only for the larger societies does the value of $p_i = .40$ even approach the point at which the probability of an intransitivity begins to decrease. Yet, once this probability begins to drop, the fall is precipitous, particularly in comparison to the gradual rise in the probability of a cyclical majority for smaller values of p_i the probability of a transitive indifference (i.e., strong or weak) occurring increases, albeit not always monotonically given the undersized number of samples, with p_i. For a three-member society, however, this probability does not reach beyond one-half until fully two-thirds of all dyadic responses favor the indifference relation. For a 75 member society, on the other hand, the half-way point is reached in the interval (.45, .50) for p_i, whereupon the probability of a transitive indifference reaches unity for a p_i of only .70. But note that moderate indifference and pseudo-weak indifference are treated as intransitive outcomes (together with the paradox, or course) in Table V.

Thus, except for the smaller societies, the probability of an intransitive outcome is generally quite small except at these critical points. For small values of p_i, the most likely result (not shown in Table V) is a societal choice; for larger values, transitive indifference is virtually assured. But, if the indifference relation is here considered to require transitivity at the collective as well as individual level, we interpret the transitive I relation to indicate that members may find the various alternatives equally attractive and the intransitive I to indicate that the members find all equally distasteful. Thus, we treat intransitive I relations similarly to the paradox. But the upshot of the probabilities in Table V is that except in the area of the critical point(s), the probability of an intransitive result (either I-intransitivity or a paradox) is very small. It thus appears unlikely that a social choice function will produce a result that is distasteful to most of society. If one wants to adopt a more extreme position and treat *all* I relations as reflecting equally bad alternatives, then if p_i (and, hence, also the corresponding probabilities for the transitive indifference outcomes) is large, the question should be whether the agenda of alternatives is satisfactory. In this case, we should perhaps adjust our concern to how the alternatives being considered yet so unpopular got on the agenda, indeed to dominate it, in the first place.

Consider, now, what happens when we relax the requirement that the collective I relation be transitive. Here we assume that the alternatives over which the collective indifference relation holds are deemed to be equally desirable among the members of a society. The results in Table VI, which treat all forms of indifference (strong, weak, moderate, and pseudo-weak) as yielding a societal I relationship and all paradoxes resulting from a violation of

I-transitivity as choices indicate that the probability of a social choice is markedly increased for even moderate values of p_i. However, as might be expected, the probability that a choice will be obtained decreases as the size of the group increases, except in the areas around the critical point (a) *and* for small values of p_i (where the differences across societies are minute). Yet, even for a society with 75 members and a probability of individual indifference of .40, we still have a .640 probability of a social choice.

Note that the probability of the paradox drops significantly (not presented in the table), since we here assume that the probability is a constant equal to the value when $p_i = 0$. Recall from note 3 above that even this value of the paradox for $p_i > 0$ overstates substantially the true value of the paradox. So, treating the I relation as intransitive over the collectivity, we can dramatically increase the probability of a social choice. In the vicinity of the critical point(s), however, the I relation begins to dominate the choice outcome, somewhat slowly for the smaller societies, but much more rapidly for the larger ones. Indeed, for the 75 member society, a $p_i = .333$ yields the probability of societal indifference of only .1000; for $p_i = .45$, this probability has increased to .512 and is virtually unity when $p_i = .55$. In contrast, the three-member society shows a much more pronounced rate of increase for smaller values of p_i and a correspondingly lower rate for larger values: Not until $p_i = .667$ does the probability of societal indifference exceed seventy percent; the ninety percent barrier is not crossed until $p_i = .85$. Yet, the values for the probability of the collective I outcome to: (1) dominate the choice relation, and (2) exceed one-half seem to be *relatively* constant over the set of societies. In each case, these results occur at the previously identified critical point(s).

Thus, even though the I relation is not required to satisfy transitivity at the level of collective choice, we still have large probabilities on societal indifference — as we might have expected given Mahoney's result. If we assume that all alternatives are equally desirable, then such a situation may not be very disturbing to the democrat whose main objective is to find a social decision function which produces results which are neither imposed nor dictatorial. After all, if a voter finds all alternatives equally attractive, then is he not likely to be content with whatever outcome is ultimately selected ?

The problem is not easily resolvable because of the lack of analytic results for partial cultures which might be comparable to that of Mahoney for partial cultures. I.e., if less than a .40 probability of being indifferent between any dyad of alternatives is great enough for large societies *always* to be strongly indifferent among a set of three alternatives, what about the remaining voters who have a probability of .60 of either producing a social choice or an intransitivity at the level of collective decision-making or who, at the individual level, have this same probability of either preferring alternative a to b or *vice versa* ? Is this result not somewhat perverse, even if we relax the collective I-transitivity assumption ? To avoid it, need we not also relax the individual I-transitivity assumption ? We are thus potentially led into an other kind of paradox which seems to overshadow the various other kinds of indifference (weak, moderate, and pseudo-weak) discussed above : May the individual indifference probability's "power" to yield collective strong indifference even for moderate levels of p_i (i.e., less than .50) be too overwhelming, so that either a paradox is introduced (in the region of the critical point) or strong indifference dominates when we might otherwise be able to generate a social choice ? For an impartial culture, the reasuring aspect of Mahoney's finding is that it does occur at a value greater than one-third (equiprobability for each individual outcome, aP_ib, aI_ib, bP_ia). But note how close to one-third the asymptotic critical point is. As the number of issues increases and as we depart from an impartial culture, how close might we come (or might we actually arrive at) an imposed or dictatorial solution ? The paradox of voting as we have traditionally examined it *may* not be very critical for systems permitting individual indifference, but it does raise associated questions of the stability and desirability of "democratic" decision rules under alternative assumptions about transitivity.

NOTES

[1] This work is based upon a previous paper (Uslaner, 1974) in which work on the paradox of voting allowing for individual indifference was begun. Since that time, the conceptual framework in that paper has been largely through the efforts of Richard G. Smith, who received a Master's degree in Operations Research from the Department of Industrial and Systems Engineering at the University of Florida in 1974 and John Mahoney, a member of that department. Smith originally sought to convert my simulation program into another language; in the process, we both developed an awareness that the logic in the earlier paper needed to be considerably refined. The results presented in this paper were derived

by Smith, in part for his thesis (Smith, 1974) and in part for me. The conceptual refinement was a joint effort, although I owe no small debt to him. The present paper is quite different from his essay in all but the most tangential ways in the discussion of the impact of indifference. Furthermore, the logic behind Smith's program closely followed that of the one I wrote for the earlier paper. Independent of Smith's essay, Mahoney began to work on some formal properties of voting bodies suggested by this very problem; see Mahoney (1974). I have drawn on his unpublished results only insofar as they are cited in the text. So, while many of the simulation runs discussed herein are those of Smith, my treatment of them is different from his and particularly my aggregation decisions in Tavles V and VI below. Indeed, not every result I wanted was available in the summary sheets of his results which he provided me (at my request) and many of which do not appear in his essay. Cf. note 3 below.

[2]Nevada adopted a similar plan recently. See *State Government News* (January, 1976) : 15.

[3]Unfortunately, the results of pseudo-paradoxes were not available for examination here (cf. note 1 above). Thus, I made an assumption which is bound to underestimate the true probability of a social choice; I assumed that the value of the probability of the paradox was a constant equal to the probability when $p_i = 0$ and subtracted larger values of the probability from that. If the simulated values were smaller, no additions were made to the social choice estimate. This overestimates the probability of the paradox since it assumes that paradoxes involving only strong orders are much more likely to occur than those involving weak orders. Future simulations will investigate pseudo-paradoxes as well. Also note that my use of the term "type 1 paradox" for a pseudo-paradox is only partially consistent with Klahr (1966). We do have a paradox under I-transitivity, but not at all if we relax this restriction. Thus, unlike Klahr's result in which there is an intransitivity among all but one alternatives, the dilemma of an intransitivity here depends upon the very nature of the transitive relation itself.

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