CHURCH'S THEOREM AND THE ANALYTIC—SYNTHETIC DISTINCTION IN MATHEMATICS

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Kant's classification of mathematical truth as synthetic a priori has given rise to a very considerable literature on the subject. We will limit ourselves here to discussing a recent attempt to vindicate Kant in terms of certain results in contemporary first order logic. In a series of lectures delivered at Oxford in the early sixties, J. Hintikka has proposed formal explications of the Kantian distinction between analytic and synthetic truths and arguments. Entitled "An Analysis of Analyticity", "Are Logical Truths Tautologies ? ", "Kant Vindicated", and "Kant and the Tradition of Analysis", these lectures now form a central part of Hintikka's book *Logic, Language Games and Information : Kantian Themes in the Philosophy of Logic* (Oxford, 1973); henceforth references to Hintikka's presentation will simply be given by the appropriate page number in the book. We shall first give an overview of Hintikka's arguments, then examine them criticallly, and finally argue for Church's Theorem as a superior formal vindication of Kant's position.

Hintikka's explications of Kant

At the outset, Hintikka understands Kant as holding an argument to be analytic if its conclusion is obtained merely by "analyzing" the premises, that is, if its conclusion expresses something which is already contained or thought of in the premises. In particular, an argument step would be analytic for Kant if one cannot conceive of the premises "clearly and distinctly" without already thinking of the conclusion (p. 124). Hintikka rejects as unfaithful to Kant's view the identifying of analytic truth with conceptual truth, however, on the

grounds that Kant explicitly considered to be synthetic several truths obtained quite conceptually, by using logical laws and definitions exclusively.

Opining that the most concrete content of a given set of premises is a certain number of interrelated individuals or entities to be taken into consideration, Hintikka instead proposes that Kant wished to restrict his notion of analyticity only to those conceptual truths obtained without introducing into the discussion a greater number of individuals than that presented by the premises. Hintikka esteems that this sense of analytic truth, which rests on the analysis of "concrete complexes of individuals", is clearer than one which would rely on the analysis of concepts, "to the extent the notion of an individual object can be understood better than that of a general concept" (p. 136), and backs up his "objectual" interpretation of analyticity by recalling that Kant considered interindividual inferences concerning existence to be impossible by analytic means.

More explicitly, Hintikka defined the *degree* of a sentence expressed in a first-order language as the sum of the number of free singular terms and of the maximal number of quantifiers with intersecting scopes occurring in the sentence. This notion can be sharpened to account even more precisely for the number of individuals or "objects" which a given sentence invites one to consider in relation to each other (p. 142), but for our present purposes the above definition will suffice. Hintikka first suggests that a *synthetic a priori* argument step may be explicated as one which adds to the number of individuals considered in relation to each other in the premises, by producing a conclusion of higher degree than that of each of the premises. An argument step would then be *analytic* if the degree of its conclusion were less than or equal to the degree of at least one of its premises. By extension, an argument involving several steps would be analytic only if each of its argument steps were analytic, and synthetic otherwise.

Such an explication would be unsatisfactory, however, as according to it an argument may well be analytic while its contrapositive, obtained by contraposing each of its steps, may be synthetic. Hintikka himself finds this "rather odd", and consequently proposes a "much more important" and "more interesting" explication : "a proof of S_2 from S_1 is *analytic* if the degree of each of the intermediary stages is smaller than or equal to the degree of either S_1 or S_2" (p. 143). Under this explication of the analytic-synthetic distinction, the proof of S_2 from S_1 and the contrapositive proof of *not*-S_1 from *not*-S_2 will be either both analytic or both synthetic simultaneously.

Hintikka's basic idea is thus to explicate "conceptual analysis" as a method of argument which is limited by a certain order of complexity, determined by the number of interrelated individuals to be taken into consideration in carrying out the argument. Kant described the essence of mathematical proof, which he judged to be synthetic in nature, as relying on the use of *constructions,* that is, on the introduction or exhibition a priori of an intuition corresponding to a general concept. A common interpretation of what Kant meant by this, held for example by Russell, is that in a geometrical argument one must appeal to the constructed figure in order to deduce its properties by some sort of geometric intuition or imagination. In contrast to this, Hintikka maintains that Kant meant by "intuition" simply a "representative of an individual to illustrate a general concept" (p. 145), and so simply meant by "construction" the "introduction of a representative of an individual". Mathematical proofs would then be generally synthetic in the sense that in the course of the proof one can be lead to increasing the number of interrelated representatives of individuals, or "values of variables", under consideration (p. 137).

According to Hintikka, Kant would not even have considered as logic what we are at present accustomed to call the logic of quantification. Typically quantificational modes of inference such as existential instantiation would have been in Kant's eyes mathematical (and synthetic) rather than logical (and analytic), for quantification theory is centered on "intuitive" methods, following Hintikka's explication of Kant's terminology (p. 140).

To establish that mathematical argument is synthetic a priori in nature, under this first somewhat unusual interpretation, Hintikka shows that degree-increasing arguments are indispensable in the development of mathematical theories. Hintikka's demonstration of this resides essentially in that, if an analytic (degree-respecting) proof existed for each provable mathematical result, then a decision procedure would exist for the mathematical theory, which, by Church's Theorem, is generally not so. Without going into the fine detail of Hintikka's argument (pp. 180 ff.), it is sufficient to observe that the existence of an analytic proof for a given statement would essentially enable one to apply an upper bound, determined by the degree of the statement to be proven, in seeking out the proof candidates for the statement.

Hintikka adds further support to Kant's evaluation of mathematical argument by offering a second vindication in terms of the more widespread interpretation, that arguments are analytic if they convey no information which was not already contained in the

premises. This interpretation identifies analytic arguments with tautologies. By means of his notion of *distributive normal form,* which is a generalisation of the well-known disjunctive normal forms of first-order logic, Hintikka proposes two measures of the amount of information which a sentence may be said to contain, one of which he calls the *surface information* given by the sentence, the other being a measure of its *depth information* (pp. 187 ff.). We will limit ourselves here to noting that Hintikka's surface information can be obtained directly from the sentence's distributive normal form, whereas its depth information is defined via a limit process leading through successive expansions of its normal form into constituents of continually increasing degree. Hintikka's second vindication of Kant lies essentially in the observation that under this information-theoretic interpretation of analyticity of an argument, the surface information given by the conclusion may be greater than that contained in the premises, even though the depth information of conclusion and premises must be equal if the argument is logically valid.

Hence if one takes the "meaning" of a mathematical statement formulated in a first-order language to be measured by its depth information, then all mathematical arguments are tautological, and so analytic, whereas if one takes instead Hintikka's surface information as measure of what the statement "means", then mathematical arguments are often synthetic. The essential difference between the measures of surface and of depth information lies in the decidability, or calculability, of the former and on the undecidability of the latter. One cannot determine beforehand a degree (in Hintikka's sense) at which the expansion process of a given sentence's distributive normal form will yield its total depth information, once again because of Church's Theorem.

We have seen that both of Hintikka's approaches to a vindication of Kant rest essentially on the undecidability of first-order logic, and hence on the undecidability, in general, of mathematical theories which require the expressive power of the language of first-order logic for their adequate formalization. In the first vindication offered, the necessity of introducing argument steps of higher degree than that of the premises and conclusion derives from the undecidability of the formal calculus, while in the second vindication the significance of the distinction between surface and depth information also turns essentially on undecidability. One is consequently led to surmise that Church's theorem itself may be considered as vindicating Kant more fundamentally and transparently than Hintikka's somewhat particular and technical

explications. But before developing this line of thought we point out a few other weaknesses in Hintikka's presentation.

Some difficulties in Hintikka's position

Because of the problem arising with the contrapositive of an argument, we have seen that Hintikka judges more satisfactory to explicate an argument as analytic in the sense of not introducing additional entities into consideration, if it contains no intermediate formulas of degree higher than that of each of the premises and of the conclusion. However, whether or not an argument and its contrapositive are logically equivalent or not depends on the basic logical system adopted. For instance, this particular logical equivalence does not hold for the intuitionistic propositional calculus, and one can well raise the question as to what laws of logic Kant would have allowed in an analytic argument since, as Hintikka himself observes, according to some of Kant's pronouncements a truth would be analytic only if it could be established "by the sole means of the principles of contradiction and of identity" (p. 128).

Furthermore, as van Benthen (1974) has pointed out, every one-step argument would be analytic in Hintikka's expanded objectual sense simply because of the absence of any intermediary formulas! Even more fundamentally, it would appear that analyticity, as a relation between statements, must be a transitive relation. For if one cannot conceive clearly of S_1 without at the same time conceiving of S_2, and if one cannot conceive clearly of S_2 without at the same time conceiving of S_3, then it seems imperative to conclude that one cannot conceive clearly of S_1 without at the same time conceiving of S_3. Thus transitivity would appear to be a desideratum to be satisfied by any explication of analyticity. But Hintikka's expanded objectual explication fails on this count, since given any argument from S_1 to S_3 synthetic in Hintikka's sense, each of the single argument steps making up the total argument would be analytic in this same sense, yet the overall (composite) argument would fail to be so. Or again, let S_3 be an intermediary formula of maximal degree appearing in a synthetic overall argument from S_1 to S_2 : then the portion of the argument leading from S_1 to S_3 would be analytic, as would the portion leading from S_3 to S_2, whereas the composition of these two analytic portions would be synthetic.

Hintikka's attempt to statically characterize analyticity in terms of his notion of degree therefore runs aground on technical considerations of the simplest nature. One can also object to his Quinean interpretation of Kant's notion of intuition, in terms of

introducing new "entities" or "values of variables", on the ground
that the existential interpretation of the quantifiers is not their only
possible interpretation. In Weingartner (1966), Hintikka admits that
his explication fails to stand up under the substitutional
interpretation of the quantifiers put forth by Marcus (1962).

These difficulties lead one to seek to justify Kant's distinction
through a broader interpretation somewhat along the lines of that
given by Parsons (1967), where Kant is taken to claim essentially
that we cannot carry out any fairly elaborate reasoning in
mathematics without resorting to a symbolic representation. Parsons
proposes that in the case of a "substantial" mathematical question,
before the construction of a proof or a calculation, we cannot see
any reason for the mathematical state of affairs being what it is, and
that this obligation to resort to mathematical symbolism was the
principal reason why Kant asserted that mathematics proceeds by
"representing concepts in intuition". The resorting to a proof
formulated in first-order logic is in this sense simply a formalization
of the mathematical proof already deemed necessary and carried out
at a less formal level, and the syntheticity of the argument does not
then depend on such formal contingencies as which interpretation of
the quantifiers one adopts and which syntactical laws of logic one
acknowledges.

We come thence to our main criticism of Hintikka's proposed
justifications of Kant. Entreprises such as Hintikka's, which seek to
explicate in formal logic notions or distinctions which have to do
with *how we arrive at* mathematical truths, run the risk of fossilizing
at the level of the formal syntax or so-called "semantics" of
formalized mathematical theories, for the sake of increased
"objectivity" or "exactness", philosophical notions which more
properly concern the description of mathematics as an on-going
process rather than as a fixed body of demonstrated facts. At the
heart of what Kant considered an analytic truth to be, is the
transparent, free-flowing, automatic, self-suggesting nature of such
arguments. In contrast, to convince oneself or one's auditors to
accept a synthetic statement as true, one must contribute further
clues, one must act upon the premises and carry out auxiliary,
non-immediate constructions before the truth is revealed. As Kitcher
(1975) puts it, Hintikka's explication of Kant's notion of
construction unfortunately divorces "intuition" from its epistemo-
logical role.

Hintikka himself emphasizes that Kant's distinction was inspired
by that made by the Greek geometers who distinguished five steps in
a geometrical proof : the enunciation of the proposition, its

setting-out, the construction, the proof proper, and the conclusion (p. 208). It is the setting-out and construction phases of the demonstration, when present, which Hintikka claims Kant to have considered to be synthetic, while once the construction carried out, the rest of the demonstration, the proof proper, was in Kant's eyes analytic. Hintikka cites Aristotle to fix the origin of this preoccupation with the "problematic" aspect of geometrical demonstrations, underlining that Aristotle set off deliberation as analysis and *acting* on the deliberation as synthesis : "Aristotle writes that 'it is by activity also that geometrical propositions are discovered, for we find them by dividing. If the figures had already been divided, the propositions would have been obvious'... once the suitable constructions have been carried out, the truth of a proposition 'would be evident to anyone as soon as he saw the figure' " (p. 203).

Hintikka further quotes Leibniz who "wrote in the same vein that in geometry 'the greatest art often consists in finding this preparation ... in ordinary geometry, we still have no method of determining the *best constructions* when the problems are a little complex' " (p. 203). In addition, Hintikka claims that awareness of the problematic sense of analysis was one of the basic methodological ideas of Galileo and of Newton, thus establishing that the original Greek heuristic appreciation of the analytic-synthetic distinction remained a lively tradition right up to Kant. It is through the implication of *activity* that Hintikka himself understands syntheticity : "Kant seems to have thought it is the necessity of actually carrying out arithmetical and algebraic operations that makes the resulting equations or statements synthetic" (p. 213); "A working geometer ... is likely to have learned from his experience that the discovery of suitable auxiliary constructions is frequently not only the most difficult but also the really essential part of proving a geometrical proposition" (p. 202).

In seeking to validate his first explication of Kant's distinction in terms of his notion of depth, Hintikka indeed amply stresses that necessary heuristic activity is what characterizes synthetic mathematical arguments : "If we have to do something ourselves before we can draw a conclusion, namely, to introduce new auxiliary individuals into the argument, then... logical inferences which are synthetic in this sense will not be inferences we have to draw, but rather inferences we can draw — if we want to and if we are clever enough" (p. 197). Hintikka also emphasizes that activity on the part of the demonstrator is what essentially validates his distinction between surface and depth information, and hence his second vindication of Kant in terms of his notion of surface information :

"Surface information is the information which a sentence gives us before we have done any one of the many things we can do to it by means of logic in order to bring out all the information there may be hidden in it ... it is readily seen that there is no recursive method of computing the amount of depth information a sentence carries" (in Weingartner 1966, p. 244).

Hence we may consider the problematic or heuristic sense as fundamental in Kant's analytic-synthetic distinction as Hintikka himself sees it, especially since Hintikka in a later paper (1968) decides to present his second interpretation, based on the more "problematic" distinction between surface and depth information, as a better explication of Kant's distinction than that based on his first "objectual" interpretation. It is regrettable, then, that Hintikka does not simply seek to validate the Kantian distinction on the basis of Church's Theorem, instead of ignoring the latter's evident epistemological implications in favour of his own technical results on the degree and distributive normal form of statements, derived from Church's Theorem.

That the synthetic a priori nature of mathematics rests on the non-automaticity or undecidability of mathematical proof cannot indeed be said more clearly than by Hintikka himself : "that the truths of logic are not laws of thought is best brought out by the realization that... the truths of logic do not come to us automatically but require work on our part. The rules of this work cannot be laid down completely" (p. 198). For Hintikka, the necessity of this additional, required work constitutes the essential justification of Kant's distinction between analytic judgements, which are merely explicative *(Erläuterungsurteile)*, and synthetic judgements, which extend the scope of our knowledge *(Erweiterungsurteile)*. Yet, obviously, without Church's Theorem, the distinction would simply have no objective foothold in contemporary logic.

Epistemological Consequences of Church's Theorem

We have elsewhere already argued the central importance, in an adequate epistemology of mathematics, of recognizing and presenting mathematics as an on-going practice, as a body of knowledge in constant movement sustained and regulated by a mathematical community of understanding, and expressed our preference for perceiving mathematical work as process more than as finished product (Castonguay 1972). If a significant vindication of Kant's evaluation of mathematical truths as synthetic a priori is at all to be sought in modern formal logic, it seems most appropriate to

emphasize those aspects of Kant's thought which were inspired more or less directly by the ancient Greeks' concern for mathematical heuristics, and to view Kant's position as essentially justified by Church's Theorem itself, rather than by what Moss (1971) has termed a derived "ontological" thesis.

Hintikka's vindications of Kant themselves depend for their justification and interest on Church's Thesis and Theorem, which establish more broadly and more accessibly than does Hintikka that there is no routine way in which consequences may generally be seen to flow effortlessly from premises in mathematics, or, more precisely, that there is no automatic, effective proof procedure for mathematical statements formulated in first-order logic. Little is to be gained by diverting one's attention away from this widely accepted result, and towards Hintikka's somewhat peculiar interpretation of Kant's "intuitions" or towards his particular formal explication of what a mathematical statement "means". Church's Theorem explicitly, objectively and completely puts an end to the Leibnizean dream of a logical calculus, of an automatic, autonomous decision method by which the truths of mathematics can be identified. This result is less open to debate than Hintikka's proposals in that this interpretation of Church's Theorem as well as the formal explications of the notions of decidability and computability are very widely accepted (see Berg and Chihara 1975 and Ross 1974 for some recent discussion of Church's Thesis).

Also, in view of the numerous well-known setbacks inflicted upon the quest for an absolute foundation for mathematics in recent times, Church's results appear more relevant to a contemporary vindication of Kant's distinction insofar as it can be taken to concern mathematics as an activity rather than as a static assembly of facts. The realization of the undecidable nature of mathematics is admirably well suited to the understanding of mathematics as an open-ended process, and to the description of actual mathematical practice.

As illustration of the fruitfulness of this dynamic approach to the epistemology of mathematics, one can observe the gradual formation and development of a "typical" mathematician. The student of mathematics must generally undergo a long — and at times frustrating — period of initiation to mathematical maturity, where what is obvious for the teacher who would show the way and awaken the student's perception of mathematical truths, is often far from obvious for the aspirant. The desired maturation of the student's mathematical intuition is usually obtained only through a period of several years of contact with basic mathematical theories and of

sustained effort in trying to recreate and to apply the patterns of definition and of deduction practiced by the mathematical community. Once familiarity with the current proof techniques in a given domain of mathematics has lead to sufficient mastery of the known methods of deduction, it becomes adequate to offer a token "it's obvious", "this can be proven in the usual way", or "this follows naturally" as a complete demonstration to a great number of statements, since each of the participants to the mathematical discussion know quite exactly what proof steps are required, and which "constructions" to apply.

This high degree of insight which mathematicians seek to instill in students who would make a career of mathematics is commonly referred to as "mathematical maturity" and may be taken to be the source of Kant's "mathematical intuitions", insofar as such a priori intuitions are quite removed from more common and widespread "logical" reflexes. This mathematical maturation is a nonending process, but it is sufficient that an aspirant acquire a minimal measure of it before he can be considered a full-fledged mathematician. A "mathematician" is an individual who has definitely acquired certain relatively uncommon reflexes of reasoning, upon which he hopefully will be able to further build and innovate in the future.

It is the open-endedness of this process of maturation, not only of the mathematician as individual, but also of a given field of mathematics as a body of knowledge, which renders so natural the interpretation of Kant's analytic-synthetic distinction in terms of Church's Thesis and Theorem. Once the known definitions and proof techniques proper to a given field of mathematical study, and which are so often Greek to the layman or uninitiated philosopher, have been mastered by the mathematical researcher to the point that they have become so to speak *instinctive,* the logical configurations of the particular mathematical concepts in play may consciously — and very often unconsciously, as Poincaré has eloquently attested — suggest new proofs, new procedures of demonstration which advance the knowledge of the given field *(Erweiterungsurteile),* though they in turn soon become part of the routine proof procedures which automatically come to the minds of colleagues confronted with further problems in the field. Such a fate awaits, for example, any proof of Fermat's Last Theorem, if ever such a proof is found.

In this way a revolutionary new technique of proof appears as a break-through beyond a first recursive ring of results, obtainable through already known proof procedures, into a larger recursive ring of results obtainable through the standard procedures augmented by

the new technique. The synthetic a priori nature of mathematical truths lies in this very open-endedness of mathematical knowledge, the essentiality of which rests on Church's Theorem.

Bringing Kant up to date through such an approach centred on mathematical activity appears more illuminating than Hintikka's static proposals, and not the least because the acceptance of this view necessarily implies recognition of the fact that the distinction between analytic and synthetic statements in mathematics is a relative and floating one. A relatively simple mathematical statement may be perceived as non-automatic, or synthetic a priori, by a layman, though the proof of the statement given effortlessly to him by a student in mathematics would make the statement analytic in the latter's estimation. In turn, a more complex statement may appear to the student synthetic but to the mature mathematician no more than analytic. Further yet, the auditors of a research colloquium may perceive the freshly conceived proofs of certain statements as synthetic, requiring effort on their part, whereas the iniated few who are presenting papers find them already automatic. The frontier between the automatic part of mathematics and the synthetic is thus constantly shifting.

As a consequence it seems improper to seek to answer, in the absolute, whether a given mathematical statement or argument is analytic or synthetic, obvious or interesting, trivial or imaginative. What a mathematical statement, at first glance, is considered to "mean" depends essentially on the context, on the level of mathematical maturity of the discussants and on the state of development of the subject at the time, and it is unrealistic to explicate such a fluid concept in terms of Hintikka's static notion of surface information. Nor can one usefully attempt to relativize Hintikka's explication of meaning by allowing the realm of analyticity to be extended to expansions of higher and higher degree of the constituents of a mathematical statement's distributive normal form, and by introducing between the extremes of surface and depth information measures of "analytic meaning of degree n", for the choice of the "n" at which analyticity ends and syntheticity begins would still remain totally artificial.

It is in fact a commonly held judgement, in mathematics, that the meaning of a mathematical statement is best revealed by following its proof, and it indeed not infrequently happens that a new proof of an already established statement supplants older proofs because it provides more insight into the conceptual components of the statement and their interrelations. The central importance of proof in the mathematical enterprise leads us to conclude that it is the

relative ease or difficulty with which its proof can be carried out or followed that makes the proven statement analytic or synthetic, and not the comparison of the degrees or distributive normal forms of its premises and conclusion. Whether or not substantial logical activity is required to complete a proof is simply not automatically measurable.

Whether one can vindicate Kant's apparent evaluation that "conceptual analysis" does not comprise more than monadic logic, or whether the truth of a given fixed statement can be construed as "logical" and not "mathematical", seems therefore to us to be an idle preoccupation in the light of the advances made since Kant's day in our understanding of the nature of mathematics. As to where "logic" ends and "mathematics" begins, the discussion apparently now turns on whether second-order logic is or is not set theory (Boolos 1975). Turning our minds instead with Dieudonné (1975) to what "mathematical intuition" or "imagination" means to the working mathematician, what appears to be worthwhile retaining is a dynamic, evolutional interpretation of Kant's distinction which can account for the advancing boundary of the distinction in line with the development of acquired individual or collective processes of deduction.

Such a flexible, problem-oriented approach permits the stage-by-stage integration of new proof techniques and reflexes into the set of logical tools upon which one may automatically rely in "recognizing" analytic truths and arguments. At the same time, the continuing discovery by "synthetic" methods of new logically valid proofs and statements is objectively guaranteed by Church's Theorem, wich thus establishes the globally synthetic a priori nature of mathematical reasoning as opposed to entirely analytic realms of judgement. The identification of the particular processes of mathematical proof claimed by a mathematician or philosopher to be properly synthetic may then vary as widely as from Russell's to Hintikka's.

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