A Form of Calculus

Introduction

The starting-point in A is a "system" S, in fact two connected inductive definitions, in which two classes are defined. The concept of a figure ("Figur" in German) is provisionally introduced in an informal way.

In B these two classes are generated by means of calculi. The problem of their decidability is treated. The proofs of the decidability of a calculus are based upon numbers.

In C a form of calculus is described, which generates "figures" by means of manipulations upon numbers. Such a calculus generates series by recursion and these series determine entirely the construction of the figures. The series memorizes the way a figure is constructed. In fact the series determines a P-marker, but this aspect is not treated in this paper. In an attempt to start from the base we have only tried to define the class of those calculi which generate the finite-state languages. To demonstrate the strongness of this form of calculus a class of mirror-expressions is generated.

In D a calculus of the same type, but with transformations, is constructed. Because of the abstract character of such a calculus an axiom system with similar effects is constructed. An attempt is made to specify the class of the theorems of the logic of propositions.

A. The System S

In the system S two classes, of which the elements are composed of signs occurring in the alphabet of S, are defined.

1. Alphabet: X, ′, =.
2. Inductive definition of the class V:
   i. X is an element of the class V;
   ii. if T is an element of the class V, then T′ is also an element of the class V;
   iii. only these given by i and ii are elements of the class V.

Remark : T is a variable for elements of the class V and it only appears in the language in which we refer to the elements of V (meta-language).

3. Inductive definition of the class B :

   i. X = X is an element of the class B ;
   ii. if T = S is an element of the class B, then T′ = S′ too is an element of the class B ;
   iii. only these given by i and ii in 3 are elements of the class B. 1, 2, and 3 form the system S.

The elements of the class V, namely X, X′, X″, etc,. just as those of the class B, X = X, X′ = X′, X″ = X″, etc., are "figures".

In a way of intuitive approximation a figure can be considered as a constructive precept. Consequently, it can be differentiated from the particular realization printed on paper. It can be said that an appeal to the constructive precept is necessary for a particular realization.

In 2.i and 3.i, X and X = X are respectively defined as given figures. With these figures begins the construction of other figures. The way this construction ought to be done is prescribed by 2.ii and 3.ii.

A similar procedure completed with the data about alphabet and variables is named a "calculus" by Lorenzen (1).

In consequence, a calculus is apprehended by Lorenzen, as a method to construct figures.

B. The Calculi K1 and K2

1. The calculus K1 :

   i. atoms : X, ′ ;
   ii. variable for figures : T ;
   iii.
      a) beginning of the calculus : X
      b) rule of the calculus : T→ T′.

The atoms, finite in number, form the alphabet of the calculus. In this case T is a variable for the figures of K1. Further, figure X is given. The rule T→T′ is a precept for the construction of a figure T′, in case figure T is already given. So the rule can be applied to X as well as to any figure already obtained out of X, by the application of this same rule. The initial

(1) P. Lorenzen, *Einführung in die operative Logik und Mathematik*, Heidelberg, 1955, p. 12 and ff.

figures as well as those which can be obtained out of the initial figures by applying the rule, are said to be derivable in a calculus.

2. The calculus K2 :

i. atoms : X, ′, = ;
ii. variables for figures composed of atoms X or ′ : T and S ;
iii. a) A : $X = X$
b) R : $T = S \rightarrow T' = S'$.

A and R are abbreviations standing for, respectively, the beginning and the rule of the calculus. The variables T and S of K2 belong to another type than the variable T of K1, in the sense that T of K1 was a variable for figures derivable in the calculus K1 itself, whereas T and S of K2 are no variables for figures derivable in K2.

As already mentioned in ii, they are variables for figures composed of the atoms X or ′.

In order to apply the rule of K2, we must be able to recognize a figure composed of X or ′ on the right and the left side of the atom "=".

The meaning now of the statement "F is a figure exclusively composed of the atoms X or ′, ″, however, is that F is derivable in a calculus Ki of which X and ′ are the atoms.

Consequently, calculus K2 supposes a calculus Ko.

Naturally, also variables will occur in the calculus Ko. However, these will not belong any longer to the type of variables of K2, but to the type of variables of K1. For, in this case, an appeal is merely made to the figures derivable in the calculus itself.

3. The calculus K0 :

i. atoms : X, ′ ;
ii. variable for figures derivable in K0 : T ;
iii.
a) A 1 : X
A 2 : ′
b) R 1 : $T \rightarrow T'$
R 2 $T \rightarrow TX$

So we can propose to modify indication ii of the calculus K2 as follows :
ii. variables for figures of K0 : T and S.

If we want to apply the rule of K2, we always ought to be able to argue out wether we are confronted with a figure derivable in K0 or not. This will be the case if K0 is decidable. But the problem is not completely solved yet.

Both expressions

"ii. variables for figures composed of X or '" and
"ii. variables for figures of K0" must completely cover each other.

In the first expression is given that all figures exclusively composed of X or ' are considered. So all figures composed of X or ' must be derivable in K0.

Let us name the number of atoms, which occur in a figure, the length of this figure. Now, with two different signs, 2^n combinations of length n are possible. (2^n can be formulated more precisely as the cardinal number of the set of applications of a set with cardinal number n in a set with cardinal number 2 (2).

We say now that all figures of length n are derivable, if their number for m atoms amounts to m^n.

If m^n figures are derivable for each n with m atoms, we can say that all figures composed of m atoms are derivable.

The following now applies for K0 :

i. all figures of length 1 are derivable.

The number of these figures amounts indeed to $2^1 = 2$. Two figures of length 1, namely the initial figures X and ', are now derivable in K0.

ii. In order that all the figures composed of X or ' should be derivable, we still have to prove that, in case all the figures of length n are derivable, this is also the case with all the figures of length n + 1.
We suppose that the assertion for figures of length n is proved. Consequently, the number of these figures amounts to 2^n.
It is possible to apply the rules 1 and 2, separately, to each of these 2^n figures. In this way we acquire a number of figures of length n + 1. This number amounts to $2^n \times 2 = 2^{n+1}$. However, this is the number of all figures of length n + 1.

So we come to the conclusion that all the figures composed of X and ' are derivable in K0.

From this also proceeds that K0 is decidable.

Let us name W the class of figures derivable in K0. In that case, K0 is decidable if the class W and the complement of W, namely CW, are enumerable (3). A class generated by a calculus is always enumerable. This has been proved by Hermes (4) for rule systems, and a calculus can be considered thus. The complement of W, however, is the zero class which is considered as enumerable (5). Consequently, K0 is decidable.

(2) K. Kuratowski, *Introduction to Set Theory and Topology*, Oxford, 1961, p. 70.

(3) H. Hermes, *Aufzählbarkeit, Entscheidbarkeit, Berechenbarkeit*, Heidelberg, 1961, p. 17.

(4) Ibid. p. 15.

(5) Ibid. p. 16.

In illustration we will give a similar demonstration for K1.

4. K1 is decidable

We name V the class of figures derivable from K1 and CV the complement of V. The complement of V must be determined in respect of a defined class. As in K1, only figures composed of X and ′ are derivable, we shall determine the complement of V in respect of the class W of K_0, which includes all figures being composed of these atoms.

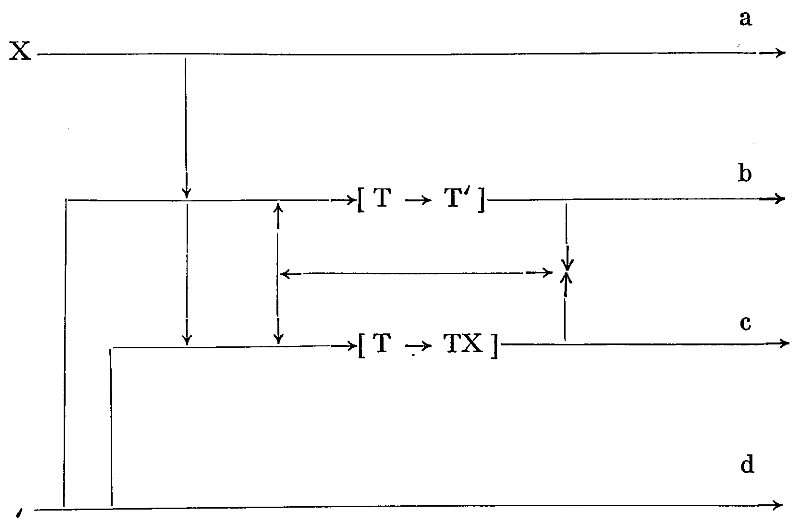
We may conclude now that the class V is enumerable, as it is generated by a caculus K1. CV will be enumerable too, if we succeed in determining a calculus for this class as well. If this is the case, we may conclude to the decidability of K1.

V and CV are complementary in respect of W, on condition that W is identical to the sum of V and CV while these two classes have no common element. The calculus for CV will be formed thus that it meets both requirements.

We shall now build up two different versions for calculus K0, successively. In both versions K0 will be represented in the shape of a scheme, after the example of the combinatory structures (6).

The first version directly produces the class V and it can easely be recognized as being equivalent to the original form for K0. In the second version the classes V and CV are generated separately and form together the class W.

a) first version of K0 :

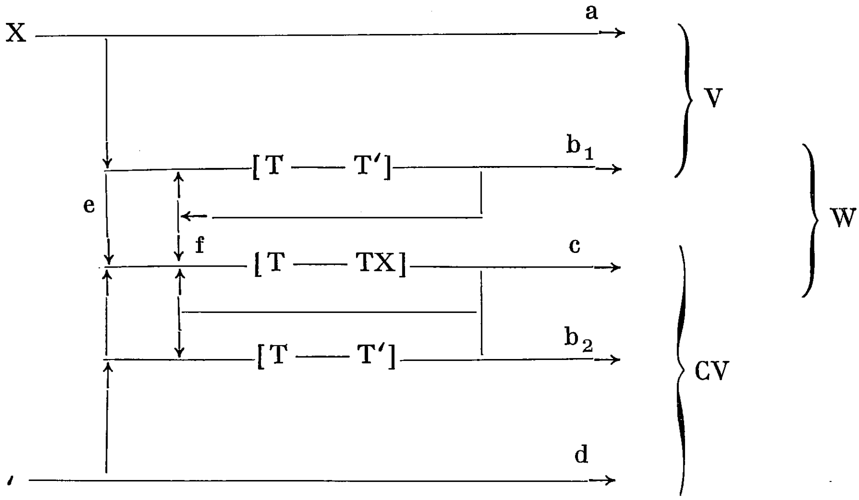


(6) M. Gazalé, *Les structures de commutation à m valeurs et les calculatrices numériques*, Louvain, 1959, p. 10.

X and ′, as the initial figures, are the imputs of the schematic calculus. They are directly referred to the "output" by the arrows a and d. Consequently, they are immediately derivable.

The arrows b and c refer the constructions obtained by applying rules 1 and 2 to the outputs. The possible "output figures" are directed to rules 1 and 2 by the other arrows in order to be submitted to their action.

b) second version of K0 :



Also from this schematic calculus can be proved that all figures composed with X or ′ are derivable in it. For the original version of K0, this proof was mainly based on the fact that R1 as well as R2 can be applied to any figure. In the above schematic version, every output-arrow is put in communication with the imput-arrows of R1 as well as of R2. Consequently, this version is equivalent to the original.

Presently, this means that W is the class of figures derivable in the second version.

Along the arrows a and b1 class V is derivated. The outputs of these are indeed determined by the initial X and the rule R1.

The arrows c, b2, and d determine a class C, of which we shall try to prove that it is identical to class CV. This is so if C and V have no common element.

It is easy to prove that for each n, there is only 1 figure of length n derivable along the arrows a and b1, thus in K1. Let us now leave section K1 of the schematic calculus out of consideration and eliminate the connections e and f with the other section. This section is a calculus with ′ as the initial figure, and R2 and R1 as rules. Here as well can easely be proved that for each n, the number of derivable figures of length n amounts to 2^{n-1}.

We have to point out that this number really amounts to 2^{n-1}, if, and only if the figure obtained by the application of R1 to figure F, differs from the figure obtained by applying R2 to this same figure F. Consequently, F′ must be different from FX. This is the case if ′ differs from X, what is rather evident. This evidence, however, is susceptible of an exact formulation by making an appeal to Lorenzen's implicit definition of inequality (7).

Because of its evidence, this remark was not made in the proof of the "completeness" of calculus K0. In this case, however, it needs to be stressed on.

We shall now examine how many figures ought to be added to this 2^{n-1} for each n, if the connections e and f are taken into account.

This time we leave the connection with d out of consideration.

a) There are $2^{n-1}-1$ figures of length n, if n is equal to 1. Consequently, this makes 0 figures of length 1. In fact, c and b2 are no direct output-arrows along which an initial figure is immediately derivable. And as already stated, d, being in fact a direct output-arrow, is not considered any longer.

b) For each n, the number of derivable figures will amount to $2^{n-1}-1$, if the following can be proved: if this is the case for length n, it is also the case for length $n+1$.

We suppose this is proved for length n. Consequently, the number of these figures amounts to $2^{n-1}-1$. R2 and R1 can be applied to each of these figures: 2^n-2. But above this, there is always 1 figure of length n too, but never more than 1, which is derivable in K1 and directed to R2 along the connection e or f. This leads to the total number of derivable figures of length $n+1$ to 2^n-2+1. However, this is $2^{(n+1)-1}-1$. This, however, is the requisite number.

For each length n, the total number of figures derivable along the output-arrows c, b2, and d amounts to $2^{n-1}+2^{n-1}-1$. This is equal to 2^n-1.

Along the output-arrows a and b1, 1 figure is derivable for each length n.

We know now that the total number of figures derivable along the five output-arrows must amount to 2^n. In fact, here it is also the case if all figures differ among themselves: $2^n-1+1=2^n$. Class V and C can not have a common element, and so we come to the conclusion that C is equal to CV.

5. The decidability of K2

(7) P. Lorenzen, *o. c.*, p. 34.

To prove the decidability of K2, the same method as for K1 can be employed. The schematic calculi, however, will be much more complex in this case. From the point of view of the practical utility this method can hardly be recommended. Therefor we shall appeal to another method which has not the same generality as the first one.

We shall give a quick outline of the proof, which will be resumed in section C, in which we shall try to give a general description of simple concatenative calculi.

A derivation A in a calculus K can be described as a finite series of figures, which are either initial figures of K or figures immediately obtained from preceding figures, by means of a rule R, belonging to K. So a derivable figure is the last figure of a derivation. Consequently, F is a figure derivable in K, if, and only if a derivation for F exists in K.

The set of all derivations in K is decidable (recursive).

The number of all derivations is nevertheless infinite, so, in case there is no derivation for a specified F, we must keep searching without ever coming to the conclusion that it does not exist.

At least this will be the case if we have no further data to our disposal. Suppose that the length of a derivation, namely, the number of figures which occur in the derivation is smaller than or equal to the length of the derivable figure. In that case only a finite number of derivations are to be considered for a specified figure F.

After having carried out a finite number of prescribed manipulations, namely, the construction of all derivations of which the length does not exceed the length of figure F, we can argue out wether F is derivable or not.

K2 now is a calculus, as we can see by the rule occurring in it, of which the length of the derivation meets the appointed condition.

Consequently, K2 is decidable.

C. General Description for Calculi

In each calculus a class H of atoms is considered as datum. This class is named the alphabet of the calculus.

Before sarting with the actual description, we shall try to describe the "construction precept" concept for these atoms more precisely.

The atoms can be considered as vectors (for instance, with 1 and 0 as basis numbers) determined by a matrix. When a graphical representation (for instance, it can also be acoustic) is presented to be identified, it can be transposed into a vector by means of the matrix. In case two graphical representations lead to the construction of one and the same vector, they are considered as identical. On the other hand, if a vector is given, it can be interpreted through the matrix in a series of indications for a graphic realization. In that case, the vector, interpreted by means of the matrix, is a "construction precept". Consequently, H is the class of vectors for which a special matrix is given (8). So H is a subclass of the class of vectors determinable by the matrix:

$H' = \{ v_i \mid v_i \text{ is determinable by } M \}$ for $1 \leqslant i \leqslant r$,
$H \subseteq H'$,
$H = (v_1, \ldots, v_m)$ for $m \leqslant r$.

To facilitate some constructions, an appeal is made to the zero vector v_0, as well. Whereas $v_0 = (0, 0, \ldots, 0)$, not a single realization is determined along v_0 and v_0 can be considered as the identity element for the concatenation: $v_0 v_i = v_i$.

1. Construction of figures in H

The figures in H are all the possible finite concatenative combinations built up of elements of H.

A calculus determines the construction of a class of series R_n and each series R_n determines the construction of a figure F (9).

First we shall give the condition to which a R_n must fulfil and the way a figure is constructed:

i. if $H = (v_0, v_1, \ldots, v_m)$ then $h = (0, 1, \ldots, m)$;

ii. R_n is a series of n elements c_i:
$R_n = (c_0, c_1, \ldots, c_{n-1})$ for $n \geqslant 0$;

iii. each element c_i is a pair of natural numbers a_i and b_i:
$c_i = (a_i, b_i)$
— if $b_i = 0$, then $a_i = 0$ and $i = 0$,
— if $b_i \neq 0$, then $0 \leqslant a_i \leqslant i\text{-}1$ or a_i is an element of h (this is marked by $\bar{a}_i$; an exception is made for 0, where it is superfluous) then
$1 \leqslant b_i \leqslant i\text{-}1$ or $b_i = \bar{b}_i$; (thus b_i is an element of h)

iv. if $c_i = (a_i, b_i)$ then $F'_i = F'_{a_i} \frown F'_{b_i}$
— $F'_0 = F'_0 \frown F'_0 = F_0 \frown F_0 = F_0 = v_0$
— $F_0 \frown F'_j = F'_j$
— $F'_{\bar{j}} = F_j = v_j$

(8) R. J. Spinrad, *Machine recognition of hand printing*, Information and Control, 1965, p. 129 and ff.

(9) For a similar determination by a R_n: N. Bourbaki, *Théorie des Ensembles* 4, Paris, 1957, p. 9.

The sign "$\frown$" represents the operation of concatenation which will be reproduced by writing the v_i immediately next to each other in order to facilitate the expressions.

Examples :

a) $R_4 = ((0, 0), (0, \bar{1}), (1, \bar{2}), (2, \bar{2}))$

$c_0 = (0, 0)$ and then $F'_0 = v_0$

$c_1 = (0, \bar{1}) \qquad F'_1 = F'_0 \frown F'_{\bar{1}} = F'_{\bar{1}} = F_1 = v_1$

$c_2 = (1, \bar{2}) \qquad F'_2 = F'_1 \frown F'_{\bar{2}} = F_1 \frown F_2 = v_1 v_2$

$c_3 = (2, \bar{2}) \qquad F'_3 = F'_2 \frown F'_{\bar{2}} = F_1 \frown F_2 \frown F_2 = v_1\, v_2\, v_2$

R_4 determines the construction of the figure $v_1\, v_2\, v_2$.

b) $H = (v_0, v_1, v_2)$ and $v_1 = X$ and $v_2 = {}'$

$R_4 = ((0, 0), (0, \bar{1}), (\bar{2}, \bar{2}), (1, 2))$

$c_0 = (0, 0) \qquad F'_0 = v_0$

$c_1 = (0, \bar{1}) \qquad F'_1 = F'_0 \frown F'_{\bar{1}} = F'_{\bar{1}} = F_1 = v_1 = X$

$c_2 = (\bar{2}, \bar{2}) \qquad F'_2 = F'_{\bar{2}} \frown F'_{\bar{2}} = F_2 \frown F_2 = v_2\, v_2 = {}''$

$c_3 = (1, 2) \qquad F'_3 = F'_1 \frown F'_2 = F_1 \frown F_2 \frown F_2 = v_1 v_2 v_2 = X''$

One and the same figure, in this case $v_1 v_2 v_2$, can be determined by two different R_n. Partial, this is due to the fact that the associativity of the concatenation has been taken into account :

$$(v_1 v_2) v_2 = v_1\, (v_2 v_2).$$

Apparently, c_0 is a superfluous element. But for some series it is saving us a normalization process, on which we shall not digress any further, and it facilitates the setting up of a certain class of calculi. Consequently, the calculus for figures in H will look as follows :

i. $H = (v_0, v_1, ..., v_m)$ and $h = (0, 1, ..., m)$

ii. $c_0 = (0, 0)$

$c_{n_1+1} = (n_1, \bar{1})$

$c_{n_2+1} = (n_2, \bar{2})$

....

$c_{n_m+1} = (n_m, \bar{m})$

for $n_1, n_2, ..., n_m \geqslant 0$

The indices 1, 2, ..., m play no part in the determination of the succession of the process at the setting up of a series R_n, and were only introduced to explain the structure of the calculus. As $n_i \geqslant 0$, each n_j+1 can act as a n_i.

The determining of a series R_n, for instance $R_5 = ((0, 0), (0, \bar{2}), (1, \bar{3}), (2, \bar{7}), (3, \bar{1}))$, takes place as under:

iii. $c_0 = (0, 0)$

$c_{n_2+1} = (n_2, \bar{2})$

$c_{n_3+1} = (n_3, \bar{3})$

$c_{n_7+1} = (n_7, \bar{7})$

$c_{n_1+1} = (n_1, \bar{1})$

For $n_2 = 0$, $n_3 = n_2+1 = 1$, $n_7 = n_3+1 = 2$, $n_1 = n_7+1 = 3$, in the course of which it was supposed that $m \geqslant 7$, follows that:

$c_0 = (0, 0)$
$c_1 = (0, \bar{2})$
$c_2 = (1, \bar{3})$
$c_3 = (2, \bar{7})$
$c_4 = (3, \bar{1})$

Further it is easely seen that R_5 determines the construction of the figure $v_2v_3v_7v_1$:

R_n indicates the length of the derivation for the derivable figure F'_{n-1} and the length of this figure at the same time. Both are equal to n–1 if we leave c_0 out of consideration.

2. Construction of the figures of Lorenzen's calculus Ki, with more than one atom as initial figure.

The class H is ordered to facilitate the reference to atoms. This ordering, however, is not essential. Consequently, H and h will be considered as classes in order to determine the class h′, a subclass of h. h′ is the class of indices of the elements of H, which are considered as initial figures in the sense of Lorenzen. Such calculi are formed as under:

i. $H = (v_0, v_1, \ldots, v_m)$ and $h = (0, 1, \ldots, m)$

$h' \subseteq h$

i′ is an element of h′

Now, if v_1 and v_2, for instance, are considered as initial figures and $T \rightarrow Tv_2$ as the only rule of Lorenzen's calculus, then we acquire the following:

ii. $c_0 = (0, 0)$

$c_1 = (0, i')$

$c_{n+1} = (n, \bar{2})$ for $n \geqslant 1$

For $i' = 1$, R_4 determines the figure $v_1v_2v_2$:

iii. $c_0 = (0, 0)$

$c_1 = (0, \bar{1})$ $\quad F'_1 = F_0 \frown F'_{\bar{1}} = F'_{\bar{1}} = F_1 = v_1$

$c_2 = (1, \bar{2})$ $\quad F'_2 = F_1 \frown F'_{\bar{2}} = F_1 \frown F_2 = v_1 v_2$

$c_3 = (2, \bar{2})$ $\quad F'_3 = F'_2 \frown F'_{\bar{2}} = F_1 \frown F_2 \frown F_2 = v_1 v_2 v_2$

3. The generation of the class B (calculus of Lorenzen K2)

i. $H = (v_0, v_1, v_2)$ and $v_1 = X$, $v_2 = '$, $v_3 = \doteq$

$h = (0, 1, 2)$ and $h' = (1)$

ii. $c_0 = (0, 0)$

$c_1 = (0, \bar{1})$

$c_{n+1} = (n, \bar{2})$ for $n \geqslant 1$

$c_{j+1} = (j, \bar{3})$ for $j \geqslant 1$

$c_{j+2} = (j+1, j)$

$c_j + 3 = 0$

Before passing on to the general description of a simple concatenative calculus, the construction method ought to be completed with the operation of the converse.

4. The operation of the converse

i. if $i = \check{j}$ then $F'_i = \check{F}'_j$

ii. $\check{F}'_0 = F_0$ and $\check{F}'_{\bar{j}} = F_j$

iii. if $F'_i = F'_{a_1} \frown F'_{b_i}$ then $\check{F}'_i = F'_{a_i} \check{\frown} F'_{b_i} = \check{F}'_{b_i} \frown \check{F}'_{a_i}$

Mirror figures can be formed by adding this operation to the construction method : $v_1 v_1$, $v_2 v_2$, $v_1 v_2 v_2 v_1$, $v_2 v_1 v_1 v_2$, etc.

The calculus, which determines the construction of the class of mirror figures composed of elements of a certain class H, looks as follows :

i. $H = (v_0, v_1, \dots, v_m)$

$h = (0, 1, \dots, m)$ and $h = h'$

ii. $c_0 = (0, 0)$

$c_1 = (0, \bar{1}')$

$c_{n_1+1} = (n_1, \bar{1})$ for $n_1 \geqslant 1$

$c_{n_2+1} = (n_2, \bar{2})$ for $n_2 \geqslant 1$

....

$c_{n_m+1} = (n_m, \bar{m})$ for $n_m \geqslant 1$

$c_{p+1} = (p, \check{p})$ for $p \geqslant 1$

$c_{p+2} = 0$

For $H = (v_0, v_1, v_2)$ we are now giving the construction of $v_1 v_1 v_2 v_2 v_1 v_1$:

iii. $c_0 = (0, 0)$

$c_1 = (0, \bar{1})$ $\quad F'_1 = F_0 \frown F'_{\bar{1}} = F'_{\bar{1}} = F_1 = v_1$

$c_2 = (1, \bar{1})$ $\quad F'_2 = F_1 \frown F'_{\bar{1}} = F_1 \frown F_1 = v_1 v_1$

$c_3 = (2, \bar{2})$ $\quad F'_3 = F'_2 \frown F'_{\bar{2}} = F_1 \frown F_1 \frown F'_{\bar{2}} = F_1 \frown F_1 \frown F_2 = v_1 v_1 v_2$

$c_4 = (3, \check{3})$ $\quad F'_4 = F'_3 \frown \check{F}'_3 = F_1 \frown F_1 \frown F_2 \frown F'_2 \frown \check{F}'_{\bar{2}} =$

$= F_1 \frown F_1 \frown F_2 \frown \check{F}'_{\bar{2}} \frown \check{F}'_2 = F_1 \frown F_1 \frown F_2 \frown F_2 \frown \check{F}'_2 =$

$= F_1 \frown F_1 \frown F_2 \frown F_2 \frown F_1 \frown \check{F}'_{\bar{1}} =$

$= F_1 \frown F_1 \frown F_2 \frown F_2 \frown \check{F}'_{\bar{1}} \frown \check{F}_1 =$

$= F_1 \frown F_1 \frown F_2 \frown F_2 \frown F_1 \frown F_1 = v_1 v_1 v_2 v_2 v_1 v_1$

$c_5 = 0$

Only the figures for which c_{p+1} was determined, are considered. Consequently, we only obtain a figure in case the process stops by $c_{p+2} = 0$. So there are calculi in which ought to be stated that c_i can end a R_n.

5. Description of a simple concatenative calculus.

A simple concatenative calculus is a collection of entities: H, h, h′, R, through which a class F is generated if it meets the following requirements:

i. H is a series of elements v_i, which are determined by a matrix of the type already described:

ii. if $H = (v_0, v_1, \ldots, v_m)$ for $m \geqslant 0$, then $h = (0, 1, \ldots, m)$ and $h' \subseteq h$;

iii. $R = R_t = (C^1_s, \ldots, C^t_s)$ so that for $t \geqslant 1$

– $C^1_s = (c_0, \ldots, c_s)$ for $s \geqslant 0$,

– $C^j_s = (c_{n+1}, \ldots, c_{n+s})$ for $s \geqslant 1$ and $1 < j \leqslant t$

iv. $c_0 = (0, 0)$

$c_1 = (0, \bar{\imath}')$ and $\bar{\imath}'$ is an element of h′

$c_i = (a_j, b_j)$ for $1 < i \leqslant n+s$ and for

$D = \{ h \cup (0, \ldots, i\text{-}1) \cup (\acute{0}, \ldots, \check{\imath}\text{-}1)$

a_j, b_j are elements of D

or $c_i = 0$ for $i = s$ and $t = 1$ or $i = n+s$ and $t > 1$

v. if $c_i = (a_j, b_j)$, then $F'_i = F_{a_j} \frown F'_{b_j}$

– $F'_j \frown F_0 = F_0 \frown F'_j = F'_j$

– $F'_{\bar{j}} = F_{\bar{j}} = v_j$ for $\bar{j}$ element of h and $\bar{j} = 0$

vi. if $c_{i-1} = (a_j, b_j)$ and $i\text{-}1 = \check{k}$, then

$F'_{i-1} = \check{F}'_k = F'_{a_j} \frown F'_{b_j} = \check{F}'_{b_j} \frown \check{F}'_{a_j}$

– $\check{F}'_{\bar{j}} = F'_{\bar{j}}$

vii. the class F is the class of F'_i, with $F'_i = F_{\bar{j}_1} \frown ... \frown F_{\bar{j}_n}$ and $\bar{j}_i$ is each time an element of h, so that i = s for t = 1 or i = n+s (c_{n+s} of C^j_s) with explicit mention, for $1 < j \leqslant t$.

Now it can easely be demonstrated that the length of a derivation, namely the index of R_n less one, or n-1, is always smaller than or equal to the length of the derivable figure. Consequently, each simple concatenative calculus is decidable.

6. The generation of "regular languages".

The class of regular languages is the class of languages which can be "generated" by means of "representing expressions».

An exact formulation can be found in N. Chomsky and G. Miller 1958 and 1963 (10).

For the construction of representing expressions out of given representing expressions, in the last resort from some atoms, an appeal can be made to two operations, namely, the product operation and the "star" operation. Let us consider v_1 and v_2 as the given representing expressions; in this case, the representing expressions v_1v_2 and $(v_1,v_2)^+$ were formed from v_1 and v_2, respectively, by the product operation and the "star" operation. v_1v_2 represents only itself, while $(v_1v_2)^+$ represents the class of figures, generated by a diagram, which generates all combinations consisting of v_1 and v_2.

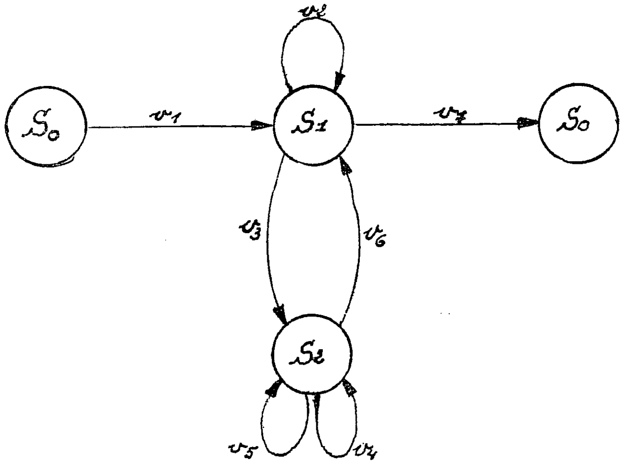
So the representing expression v_1 $(v_2,v_3(v_4,v_5)^+v_6)^+v_7)$ represents (or "generates") the class of figures, which are generated by the undermentioned "diagram". This class can be considered as a regular language L and the "diagram" as a graph of a "finite automaton".

v_1v_7, $v_1v_2v_2v_7$, $v_1v_2v_3v_4v_5v_5v_6v_2v_7$, etc..., for instance, belong to the generated figures. In order to obtain these figures, we must start from S_0 (left) and following the arrows in the indicated direction, we end in S_0 (right); each arrow which is passed along, produces a v_i.

We will make an attempt to define the class of calculi, which generate the regular languages, as the class, whose elements are constructed by means of the operations p and s and the elements c_{1_0} .

If H is a class of atoms v_j, then $c_{1_0} = (a_i, b_i) = (a_i, \bar{j})$. For every c_i, a_i = i-I. Thus c_i = (i-I, $\bar{j}$). So $c_{I_0} = (0, \bar{j})$. The elements c_{1_0} will

(10) *Formal properties of grammars, Handbook of Mathematical Psychology,* edited by D. Luce, R. Bush, E. Galanter, , New York, 1963, p. 336.
Finite state languages, Information and Control, 1958.



be called "calculus expressions". In fact they are the smallest possible "calculus expressions".

The class of "calculus expressions" is inductively defined as follows:

i. c_{1_0} is calculus expression,

ii. if $X_1, \ldots, X_n$, for $n \geqslant 1$, are calculus expressions, then so are $p(X_1, \ldots, X_n)$ and $s(X_1, \ldots, X_n)$.

The operation p corresponds to the product operation. In fact its application to a sequence of calculus expressions results in a series with a last supplementary c_i equal to zero.

Example:

$$p(c_{1_0}, c_{1_0}, c_{1_0}) = c_{1_1}, c_{1_0+1_1}, c_{1_0+1_0+1_1}, c_{1_0+1_0+1_1+1} \ (= 0) =$$
$$c_{1_1}, c_{2_1}, c_{3_1}, c_{4_1} \ (= 0).$$

We call the "o" in c_{1_0} the "product degree". An application of p gives the next higher product degree, 1 in $c_{1_1}, c_{2_1}, \ldots$, to the first argument of p. Then p proceeds through addition of indices.

The operation s corresponds to the star operation, which corresponds to one or more cycles, beginning and ending in the same state S_i, of a diagram, as it has one or more arguments. A cycle corresponds to an operation of iteration, whereby the result may be zero, as the cycles may be omitted in the travelling from S_0 to S_0. If there is more than one cycle of the same state, their order is unspecified. Now each element $c_{n+1} = (n, \bar{j})$ can perform an iteration or can be omitted and several such elements must not be ordered.

Example :

$$s(c_{1_0}, c_{1_0}, c_{1_0}) = c^{0}_{1+n_{(1)}}, c^{0}_{1+n_{(1)}}, c^{0}_{1+n_{(1)}} .$$

As all the elements of the sequence above have the same "product degree" 0 and the same "sum degree" (1), their order is not specified.

Example :

To the representing expression

a) $v_1\,(v_2,v_3(v_4,v_5)^+v_6)^+v_7$

corresponds the calculus expression

b) 1. $p(c_{1_0}, s(c_{1_0}, p(c_{1_0}, s(c_{1_0}, c_{1_0}), c_{1_0})), c_{1_0}$

where the b_i are determined from the beginning as follows :

2. $p((a_i, \bar{1}), s((a_i, \bar{2})$;......).

In order to avoid the complexity of a full and abstract definition, we shall explain the working of p and s by means of illustrations.
First a few remarks :

i. $(i+n)+j = i+n$
$j+(i+n) = j+i+n$

ii. 0 is not an element of an argument for p,
and not for s, but if $c_j = c_{i+n_s}^{p} = 0$, then $c_{j-1} = c_{i+n_{(s)}}^{(p)}$ where (p) and (s) are equal to the least degrees of the respective type (product and sum degree), while the a_i remains of the degrees p and s.

iii. $(i+n)-I = n+i-I$
Indeed, after the end of the computation the commutation of $i+n$ into $n+i$ is made in order to determine the a_i.

iv. $p((c_{1_1}, c_{2_1}), (c_{1_1}, c_{2_1})) = c_{1_1}, c_{2_1}, c_{3_1}, c_{4_1}, c_{5_1}$ (= 0).
But : $p(c_{1+n}^{0}, c_{1+n}^{0}) = c_{1+n}^{(0+1)}, c_{1+n}^{(0+1)+(0+1)}$,
$c_{1+n}^{(0+1)+(0+1)+1}$ (= 0) $= c_{1+n}^{1}, c_{1+n}^{2}, c_{1+n}^{3}$ (= 0).
Remark that the iterations are ordered by the degrees.

As a final illustration the calculus expression of our example will be computed.

$$p(c_{1_0}, s(c_{1_0}, p(c_{1_0}, s(c_{1_0}, c_{1_0}), c_{1_0})), c_{1_0})$$

We begin with the innermost calculus expression.

a) $s(c_{1_0}, c_{1_0}) = c_{1+n}{}^0, \; c_{1+n}{}^0$

b) $p(c_{1_0}, (c_{1+n}{}^0, c_{1+n}{}^0), c_{1_0}) = c_{1_1}, (c_{1+n}{}^1, c_{1+n}{}^1), c_{2+n}{}^1, c_{3+n}{}^1 =$

$c_{1_1}, c_{1+n}{}^{1}_{(1)}, c_{1+n}{}^{1}_{(1)}, c_{1+n}{}^{2}_{(1)}, c_{2+n}{}^2 (= 0)$

As a remark note the reordening of the series in a p expression after an s expression.

c) $s(c_{1_0}, (\ldots\ldots)) = c_{1+n}{}^0, \; c_{1+n}{}^{1}_{1}, \; c_{1+n}{}^{1}_{1+n}, \; c_{1+n}{}^{1}_{1+n}, \; c_{1+n}{}^{(2)}_{(1+n)}$

d) $p(c_{1_0}, (\ldots.), c_{1_0}) = c_{1_1}, c_{1+n}{}^1, c_{1+n}{}^{2}_{1}, c_{1+n}{}^{2}_{1+n}, c_{1+n}{}^{2}_{1+n},$

$c_{1+n}{}^{(3)}_{(1+n)}, c_{1+n}{}^3, c_{2+n}{}^3 \; (= 0).$

A calculus is now formed by means of the addition of c_0 to the calculus expressions.

So we obtain the following calculus:

$c_0 = (0,0)$
$c_1 = (0,\bar{1})$
$c_{n+1} = (n,\bar{2})$
$c_{j+1_1} = (j, \bar{3})$
$c_{j+1_{n+1}} = (j_n,\bar{4})$
$c_{j+1_{n+1}} = (j_n,\bar{5})$
$c_{j'+1_{(n+1)}} = (j_n,\bar{6})$
$c_{p+1} = (p, \bar{7})$
$c_{p+2} = 0$

To simplify the expressions n^1,... have been replaced by j.... The figure $v_1v_2v_3v_4v_5v_5v_6v_2v_7$ for instance, is determined by the series R_{10}, which runs as follows:

$((0,0), (0,\bar{1}), (1,\bar{2}), (2,\bar{3}), (3_1,\bar{4}), (4_2,\bar{5})\ (5_3,\bar{5}), (6_4,\bar{6}), (7,\bar{2}), (8,\bar{7}))$. The generation of the figures is easy. The complexity of the computation of calculi is due to the fact that all the processes are analysed into operations on natural numbers.

D. A Calculus K+ with Transformations and the Logic of Propositions

In order to facilitate the understanding of the working of K+, first an axiomatic system S+ will be set up. S+ is meant as an introduction to K+ and not as an equivalent system. However its classical form makes it easier to valuate the import of K+ and to discuss and illustrate certain expressions.

I. The system S+:

1. alphabet: X , ′ , = , , , ; , (,) .
   1 2 3 4 5 6 7
2. the class U:
   - i. V $\subseteq$ U,
   - ii. if M and N are elements of the class U, then (M, N) and (M ; N) are elements of the class U,
   - iii. the only elements of U are those defined in i. and ii.

     remark:
     - a) the class V in i. is already defined in the system S;
     - b) M and N in ii. are called the components of (M, N) and of (M ; N); M is his own component.
3. the class T:
   - a) i. M = M,
     - ii. if M = N, then N = M,
     - iii. if M = N and N = P, then M = P,
   - b) i. (M, M) = M
     - ii. (M, N) = (N , M)
     - iii. (M , (N , P)) = ((M , N) , P)
     - iv. (M ; (N , P)) = ((M ; N) , (M ; P))
     - v. ((M , N) ; P) = ((M ; P) , (N ; P))
     - vi. ((M , N) ; P) = (M ; (N ; P))
     - vii. ((M ; N), M) = N
   - c) i. M , N , P are elements of the class U;
     - ii. if M is a component of P, M = N and P′ is the result of the replacement of M by N in P, then P = P′;
   - d) only those determined by a, b and c are elements of T.

From b) vii. it is clear that the expression "(M ; N)" can have the intuitive sense of "the function whose value is N when the argument is M". If this function is meant to be the implication, then the sign of identity is to strong and must be replaced by a reduction from left to right: "reduces to N" or " $\geqslant$ N".

Consequently the sign ", " can be interpreted as conjunction. But, as is clear from b) v. it can be a disjunction too. The difference between disjunction and conjunction will be made clear by means of the concept of context.

The remarks above are only anticipations. Now we will go over to the construction of the calculus K+.

II. The calculus K+:

i. $H = (v_0, v_1, \dots, v_5)$

The numbers 1 to 5 correspond to elements of the alphabet of S+, as they are numbered there. The brackets are omitted. The way of construction is always given by the series R_n.

$$
\begin{array}{lll}
 & h = (0, 1, \dots, 5) & \\
 & h' = (1) & \\
\text{ii.} & c_0 = (0\,,0) & \\
 & c_1 = (0\,,\bar{1}) & \\
 & c_{n+1} = (n\,,\bar{2}) & n \geqslant 1 \\
 & c_{v+1} = (v\,,\bar{4}) & v \geqslant 1 \\
 & c_{v+2} = (v+1,\ m) & v+2 > n,\ m \geqslant 1 \\
 & c_{w+1} = (w\,,\bar{5}) & w \geqslant 1 \\
 & c_{w+2} = (w+1\,,r) & w+2 > n,\ r \geqslant 1 \\
 & c_{j+1} = (j\,,\bar{3}) & \\
 & c_{j+2} = (j+1\,,j) & \\
 & c_{j+3} = 0. &
\end{array}
$$

The indications v+2>n and w+2>n express the fact that v and w are of an higher degree than n. We have choosen this way of indication to simplify the expressions. The result is that we can't go back to c_{n+1} after the determination of a c_{v+2} or a c_{w+2}.

We will say of a series R_n that it belongs to the class of the R^1_{n+1} if it ends with a pair of natural numbers c_{n+1}. So the class of figures determined in their construction by a series R^1_{n+1} corresponds to the class V of the system S.

Further we distinguish the classes of the R^2_{v+2}, of the R^3_{w+2} and of the R^4_{j+2}, which represent the series ending respectively with a c_{v+2}, a c_{w+2} and a c_{j+2}.

The class of figures determined by series R^1_{n+1}, R^2_{v+2} and R^3_{w+2} corresponds to the class U of the system S+. R^4_{j+2} determines a class which is a subclass of T of S+.

Finally the remark must be made that the numbers m and r in c_{v+2} and c_{w+2} under ii. above are given only if there is, respectively, a R^i_m and a R^i_r such that $I \leqslant i \leqslant 3$.

After these remarks we go over to the description of the transformations.

III. The transformations of K+.

1. preliminary definitions:

a) i. if R_i is a R^1_{n+1}, a R^2_{v+2} or a R^3_{w+2}, then R_i is determined in R_i,
   ii. if both R_j and R_i are a R^1_{n+1}, a R^2_{v+2} or a R^3_{w+2} and $R_j \subseteq R_i$ (R_j is then an initial subseries of R_i), then R_j is determined in R_i,
   iii. if R^2_{v+2} (or R^3_{w+2}) is determined in R_i, then R^i_m (or R^i_r) is determined in R_i (for the m (or r) element of c_{v+2} (or c_{w+2})).

b) R_i' is a transform of R_i if and only if R_i' is the result of the application of one or more operations t^i, called transformations, upon series determined in R_i: $T(R_i) = R_i'$.

c) to express and execute the transformations, which are a form of calculation, we need the following definitions:
   i. $R^i_{x+a} + R^j_{y+b} = R^j_{x+a+b}$
   ii. $/ R^2_{v+2} / = R^i_m$
   iii. $/ R^3_{w+2} / = R^i_r$
   iv. $R^i_x + R^i_y = R^i_x + R^2_{v+2} = R^2_{x+2}$ for $R^i_v = R^i_x$ and $R^i_m = R^i_y$
   v. if $R^i_x = R^i_y + R^i_z$ then
   $R^i_{\breve{x}} = R^i_y \breve{+} R^i_z = R^i_{\breve{z}} + R^i_{\breve{y}}$.

2. the transformations of atom 4, ", ".

These transformations are the idempotency, the commutativity and the associativity of atom 4 presented in this order.

As it is in general necessary to differentiate the R^i_m, we will number them, e.g., $R^i_{m_1}$ and write $R^2_{{v+2}_1}$ for the R^2_{v+2} whose R^i_m is $R^i_{m_1}$. This is the case when $c_{v+2} = (v+1, m_1)$.

i. $t^1(R^2_{v+2}) = R^i_v$, for $R^i_m = R^i_v$ (from (M , M) to M)
   $t^{1\prime}(R^i_x) = R^i_x + R^i_x = R^2_{x+2}$ (from M to (M , M))
   – i corresponds to 3.b)i. of S+.

ii. $t^2(R^2_{v+2}) = R^2_{\breve{v}+2}$, for $R^i_{\breve{v}} = R^i_v$ and $R^i_{\breve{m}} = R^i_m$;
   –ii. corresponds to 3.b)ii. of S+.

iii. $t^3(R^2_{v+2}) = t^3(R^i_x + R^2_{{v+2}_1} + R^2_{{v+2}_2}) = R^2_{{x+2}_3}$,
   for $R^i_{m_3} = R^i_{m_1} + R^2_{{v+2}_2}$;

–iii. corresponds to 3.b) iii, but only from (M , N) , P to M, (N , P). The other direction is accessible by means of ii.

Example:

(Z . X) is a transform of (X , (Z , X));

a) the descrition of (X , (Z , X)) in terms of series is $R^i_x + R^2_{v+2_1} = R^2_{x+2_1}$ such that $R^i_{m_1} = R^i_z + R^2_{v+2_2}$ and $R^i_{m_2} = R^i_x$.

(x is a number such that R^i_x determines the string of atoms X)

b) $t^2(R^2_{x+2_1}) = R^2_{x+2_1} = R^i_x + R^i_{m_1} = R^i_{m_1} + R^i_x = (R^i_z + R^2_{v+2_2}) + R^i_x = (R^2_{z+2_2}) + R^2_{v+2_2} = R^2_{z+2_2+2_2}$.

To control the obtained figure, we can take the absolute values of the known subseries of the resulting series:

$R^2_{z+2_2+2_2} = (R^i_z + R^2_{v+2_2}) + R^2_{v+2_2}$

The absolute value of this series is:

$(R^i_z + R^i_{m_2}) + R^i_{m_2}$ as $/R^i_z/ = R^i_z$, $/ R^2_{v+2_2} / = R^i_{m_2}$

As $R^i_{m_2} = R^i_x$, we obtain $(R^i_z + R^i_x) + R^i_x$.

c) $t^3(R^2_{z+2_2+2_2}) = t^3(R^i_z + R^2_{v+2_2} + R^2_{v+2_2}) = R^2_{z+2_3}$

for $R^i_{m_3} = R^i_{m_2} + R^2_{v+2_2} = R^i_x + R^2_{v+2_2} = R^2_{x+2_2}$.

This means that we have obtained the figure (Z , (X , X)).

d) $t^1(R^2_{x+2_2}) = R^i_x$ as $R^i_{m_2} = R^i_v = R^i_x$

e) $R^i_z + t^1(R^i_{m_3}) = R^i_z + R^i_x = R^2_{z+2_3}$,

and so $T(R^2_{x+2_1}) = R^2_{z+2_3}$. This means that we have finally obtained the figure (Z , X).

3. the transformations of atom 5, " ; ".

i. $t^4(R^3_{w+2} + R^2_{v+2}) = R^i_r$ for $R^i_w = R^i_m$.

-i corresponds to 3 b) vii of S+ in one direction: from ((M ; N) , M) to N.

ii. the other transformations are defined as a set of transformations t^j, for $j > 4$ such that

$T(S(R_i)) = T'(S(R'_i))$ where

a) T and T′ represent the application of transformations t^i, for $1 \leqslant i \leqslant 4$,

b) $s(R_i) = R_i + R^i_x$ and $s(R'_i) = R'_i + R^i_x$ such that $R^i_x = R^i_w$ and that R^i_w belongs to the R^3_{w+2} such that one of the following conditions is satisfied:

bi. $R^3_{w+2} \subseteq R_i$ or $R^3_{w+2} \subseteq R'_i$

bii. $R^3_{w+2} = \underset{a}{R^i_m}$ and $\underset{a}{R^2_{v+2}} \subseteq R_i$ or $\underset{a}{R^2_{v+2}} \subseteq R'_i$

Example :

There is a transformation from ((M , N) ; P) to (M ; (N ; P)) :

a) s((M , N) ; P) = ((M , N) ; P) , (M , N)
   t^4(((M , N) ; P) , (M , N)) = P

b) s(M ; (N ; P)) = (M ; (N ; P)), (M , N) ; after an application of t^3 the transformation t^4 can be applied twice and we obtain P.

But there is a restriction on T′ :

i. t^4 can only be applied if the R^i_m is given by means of the operation s ;

ii. after every R^3_{w+2} has been eliminated no further transformation can be applied.

These restrictions do not hold for T.

So there is e.g. a transformation from M to (M ; M), but not from (M ; M) to M.

B.

An attempt to state the theorems of the logic of propositions.

1. Context and functors.

In the expression (M ; N) we call M the context of the kernel N. So the expression with context has the form R^3_{w+2}.

If R^i_w is the context of a kernel R^i_r and for the kernel itself holds that $R^i_r = \underset{a}{R^3_{w'+2}}$, then $R^i_w + R^i_{w'}$, which is equal to R^2_{v+2} so that $R^i_w = R^i_v$ and $R^i_{w'} = R^i_m$, is the context of $\underset{a}{R^i_r}$, such that the whole expression becomes $\underset{a}{R^{3\prime}_{w+2}}$ and $R^i_w = R^2_{v+2}$. Now there exists a transformation from (M ; (N ; P)) to ((M , N) ; P). This means that $T(R^3_{w+2}) = \underset{a}{R^{3\prime}_{w+2}}$.

The whole context is determined if R^i_r differs from R^3_{w+2}.

R^3_{w+2} is a functor if there exists a transformation from the context to R^i_r. The transform of a functor is itself a functor.

Example :

In the expression ((M ; N) ; ((N ; P) ; (M ; P))) the context is (M ; N) , (N ; P) , M. If the transformation t^4 is applied twice to this context we obtain P. So the expression is a functor.

2. The theorems without negation.

Atom 5, namely " ;", represents the implication. So the usual form M → N is rendered by M ; N. Atom 4, namely ",", represents the conjunction and the disjunction. These will be differentiated in respect of the context.

So the context determines the operational value of atom 4. The axioms of the positive logic of propositions are those stated by Hilbert and Bernays in "Grundlagen der Mathematik I" 1934, page 66.

a) the axioms of implication:

i. ((M ; N) ; ((N ; P) ; (M ; P)))

This functor was already stated in the example above.

ii. ((M ; (M ; P)) ; (M ; P)).

This axiom can be generated as follows:

The expression ((M ; P) = (M ; P)) is a "theorem" of the calculus K+. Indeed it is a R^4_{j+2}. For such a "theorem" there exists always a functor, which is, practically speaking, obtained through the replacement of the "=" by the sign ";".

Now it is easy to control that ((M ; P) ; (M ; P)) is a functor. The transformation of a functor is itself a functor. So, by $t^{1'}$, (((M , M) ; P) ; (M ; P)) is a functor. From this we obtain by the transformation 3 b) vi in S+ the functor

((M ; (M ; P)) ; (M ;P)), which was stated as the second axiom for the implication.

The generation of the third axiom proceeds from the "theorem" ((M , P) = (M , P)). There are two possible forms:

a) ((M , P) ; (M , P))
(M ; (P ; (M , P))) 3 b) vi in S+

So we propose as the first form:

iii. a (M ; (P ; (M , P)))

b) ((M , P) ; (M , P))
(M , P) ; M) , ((M , P) ; P)) 3 b) iv in S+
(M ; (P ; M)) , ((M , P) ; P)) 3 b) vi in S+.

The second form is then:

iii.b ((M ; (P ; M)) , ((M , P) ; P)

Now we consider the partial functor (M ; (P ; M)).

It is, following Hilbert and Bernays, the third axiom of implication. Here too it can be accepted as an axiom if and only if the second partial function of iiib. will also be accepted as an axiom, while the central sign of iii. b, namely ",", must be a conjunction.

Indeed, all the axioms of an axiomsystem can be considered as connected by the coniunction.

iii. (M ; (P ; M))

The second partial function will be listed as an axiom of the coniunction.
B) the axioms of the coniunction:

For the introduction of the coniunction we can choose between the already stated form iii.a and the form generated in the following way:

(P ; (M , N)) = (P ; (M , N))
((P ; (M , N)) ; ((P ; (M , N)))
(((P ; M) , (P ; N)) ; (P ; (M , N))) 3 b) iv in S+
((P ; M) ; ((P ; N) ; (P ; (M , N)))) 3 b) vi in S+
iv. ((P ; M) ; ((P ; N) ; (P ; (M , N))))

The elimination of the coniunction is stated in two partial functors:

(M , N) = (M , N)
((M , N) ; (M , N))
(((M , N) ; M) , ((M , N) ; N)) 3 b) iv in S+
v. ((M , N) ; M)
vi ((M , N) ; N)

Axiom vi is the second partial functor of iii.b. In the case of axioms v and vi too the condition must be satisfied that the central sign in the real functor is a coniunction.

This problem will be examined after the statement of the axioms of the disjunction.

C) the axioms of the disjunction.

(M , N) = (M , N)
(M , N) ; (M , N)
((M ; (M , N)), (N ; (M , N)) 3 b) v in S+
vii (M ; (M , N))
viii. (N ; (M , N))

Axioms vii and viii introduce the disjunction.
Elimination of the disjunction:

((M , N) ; P) = ((M , N) ; P)
(((M , N) ; P) ; ((M , N) ; P))
(((M ; P) , (N ; P)) ; ((M , N) ; P)) 3 b) v in S+
((M ; P) ; ((N ; P) ; ((M , N) ; P))) 3 b) vi in S+

3. The differentiation of conjunction and disjunction.

As already stated, two axioms can introduce the coniunction:

iii. a (M ; (N ; (M , N)))
iv. ((P ; M) ; ((P ; N) ; (P ; (M , N))).

The context of the first is (M , N) and of the second ((P ; M) , (P ; N) , P). By t^4 this last context reduces to (M , N). So the context of the coniunction (M , N) is (M , N). From the two axioms, which introduce the disjunction it is clear that M alone or N alone is the context of the disjunction (M , N). On this basis a full examination is possible.

At the start atom 4 is a coniunction. E.g. in the expression ((M , N) ; (M , N)) the context is (M,N) and so the second (M , N) can be interpret as a coniunction. Now the expression itself is secured by the "theorem" (M , N) = (M , N).

4. Natural deduction.

If there is a transformation from M to N, then we can state (M ; N). This corresponds to the schema of the introduction of the implication. The transformation t^4 gives the schema of the elimination of the implication : from (M ; N), M to N.

The schema itself has the form :

$$\frac{(M ; N) , M}{N}$$

The context is written, as a premiss, above the line, the kernel, as the conclusion, under the line.

There is no difficulty about the other schemata. The only remark is that the coniunction is introduced as follows :

$$\frac{M , N}{(M , N)}$$

This is the original form (M , N) ; (M , N) (11).

5. A syntactical decision method for theorems of the logic of propositions.

Five transformations, namely t^1, $t^{1'}$ to t^4, are initially given. The set of all the other transformations is defined by means of these five transformations. The operation s and the two restrictions must be justified by a theory of the context.

It remains to be proved that all transformations are theorems of the logic of propositions and that all theorems are transformations. Therefore a theory of the context is necessary.

We will give one more ilustration of the method.

(((M ; M) ; P) ; P) is a theorem of the logic of propositions and a transformation. So the context ((M ; M) ; P) and the kernel P must be reducible

(11) D. Prawitz, *Natural Deduction*, Stockholm, 1965, p. 20 (schemata).

to identical configurations. This reduction was expressed by the formula $T(S(R_i)) = T'(S(R'_i))$ and two restrictions.

a) ((M ; M) ; P) , (M ; M), M
b) P , (M ; M) , M

In a and b the operation s has been applied to the context and the kernel. After the application of t^4 both a and b are reduced to the same configuration (P , M).

In the case of a partial transformation the results of the reductions are subclasses of each other. The whole transformation can be reconstructed.

In this respect the negation produces a greater difficulty.

6. The logic with negation.

In the calculus K+ the negation sign was not introduced. But we can suppose that every expression has a symmetrical element, which is his negation.

Along with the coniunction, the minimal (12) negation can be introduced by axiom iv.

x. ((P ; M) ; ((P ; -M) ; -P)))

Here -P is (P ; (M, -M)).

The context of x reduces to (M ,–M) and the kernel to (P , –P).

So we must accept that (M , –M) = (P , –P) or the possibility of the two transformations (M , –M) ; (P , –P) and (P , –P) ; (M , –M).

Now, there is a transformation from (M, –M) into (M ; M) and from (P , –P) into (P ; P). In both cases atom 4 is a coniunction. In fact (M , –M) reduces to (M , –M) and (M ; M) to M. But from (M , –M) to M there is a partial transformation, whose whole form can easely be reconstructed. Further there is a transformation of (P ; P) into (M ; M) and from (M ; M) into (P ; P). Both reduce to (M , P). So we can accept the two original transformations.

The strict negation is in fact an elimination of the negation. This negation is eliminated along with the elimination of the disjunction.

xi. ((M ; P) ; (–M ; P) ; P)

The context of xi reduces to P and the kernel, namely P, reduces to (M , –M) , P. Atom 4 in (M , –M) is a disjunction. But (M , –M) reduces to (M ; M). Even the equality of both must be accepted. Now, both P and ((M ; M), P) reduce to (M , P).

The intuitionistic negation amounts to

xii. (–M ; (M ; P))

We must accept a transformation from the context (–M , M) into P.

(12) H. B. Curry, *Foundations of Mathematical Logic*, New York, 1963, p. 257 and ff.